

## PULL-BACK MORPHISMS FOR REFLEXIVE DIFFERENTIAL FORMS

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ABSTRACT. Let  $f : X \rightarrow Y$  be a morphism between normal complex varieties, where  $Y$  is Kawamata log terminal. Given any differential form  $\sigma$ , defined on the smooth locus of  $Y$ , we construct a “pull-back form” on  $X$ . The pull-back map obtained by this construction is  $\mathcal{O}_Y$ -linear, uniquely determined by natural universal properties and exists even in cases where the image of  $f$  is entirely contained in the singular locus of  $Y$ .

One relevant setting covered by the construction is that where  $f$  is the inclusion (or normalisation) of the singular locus  $Y_{\text{sing}}$ . As an immediate corollary, we show that differential forms defined on the smooth locus of  $Y$  induce forms on every stratum of the singularity stratification. The same result also holds for many Whitney stratifications.

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## 1. INTRODUCTION

**1.1. Introduction and statement of main result.** Differential forms and sheaves of differentials are fundamental objects and indispensable tools in the study of smooth varieties and complex manifolds. It is well-known that for singular spaces, there is no single notion of “differential form” that captures all features of the smooth case. Instead, there are several competing definitions, each generalising certain aspects. The following two classes of differential forms are particularly important.

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**Kähler differentials:** Uniquely determined by universal properties, Kähler differentials are the most fundamental notion of differential form. Their universal properties imply that Kähler differentials can be pulled back via arbitrary morphisms, and closely relate them to problems in deformation theory.

The sheaf of Kähler differentials on a scheme  $X$  is denoted  $\Omega_X^p$ . As a sheaf of  $\mathcal{O}_X$ -modules,  $\Omega_X^p$  is usually rather complicated and has both torsion and cotorsion, even if  $X$  is reduced. This makes Kähler differentials rather hard to work with in many cases of practical interest.

**Reflexive differentials:** Given a normal complex variety  $X$ , a *reflexive differential* on  $X$  is a differential form defined only on the smooth locus, without imposing any boundary condition near the singularities. Equivalently, a reflexive differential is a section in the double dual of the sheaf of Kähler differentials. Denoting the sheaf of reflexive differentials by  $\Omega_X^{[p]}$ , we have

$$\Omega_X^{[p]} = (\Omega_X^p)^{**} = \iota_*(\Omega_{X_{\text{reg}}}^p),$$

where  $\iota : X_{\text{reg}} \rightarrow X$  denotes the inclusion of the smooth locus. Since the  $\mathcal{O}_X$ -module structure is comparatively simple, reflexive differentials are often quite useful in practice. Reflexive differentials arise naturally in a number of contexts, for instance in positivity results for differentials on moduli spaces. Note that the dualizing sheaf is always a sheaf of reflexive differentials,  $\omega_X = \Omega_X^{[\dim X]}$ .

*Reflexive differentials on klt spaces.* Reflexive differential forms do not generally satisfy the same universal properties as Kähler differentials. However, it has been shown in a previous paper [GKKP11] that many of the functorial properties do hold if we restrict ourselves to normal spaces with Kawamata log terminal (klt) singularities. The following theorems summarise one of the main results in this direction.

**Theorem 1.1** (Extension Theorem, [GKKP11, Thm. 1.4]). *Let  $Y$  be a normal variety and  $f : X \rightarrow Y$  a resolution of singularities. Assume that there exists a Weil divisor  $D$  on  $Y$  such that the pair  $(Y, D)$  is klt. If*

$$\sigma \in H^0(Y, \Omega_Y^{[p]}) = H^0(Y_{\text{reg}}, \Omega_{Y_{\text{reg}}}^p)$$

*is any reflexive differential form on  $Y$ , then there exists a differential form  $\tau \in H^0(X, \Omega_X^p)$  which agrees on the complement of the  $f$ -exceptional set with the usual pull-back of the Kähler differential  $\sigma|_{Y_{\text{reg}}}$ .  $\square$*

**Theorem 1.2** (Pull-back morphisms in special cases, [GKKP11, Thm. 4.3]). *Let  $f : X \rightarrow Y$  be a morphism of normal varieties. Assume that there exists a Weil divisor  $D$  on  $Y$  such that the pair  $(Y, D)$  is klt, and assume that the image of  $f$  is not contained in the singular set  $Y_{\text{sing}}$ . Then there exist pull-back morphisms of reflexive forms,*

$$d_{\text{refl}}f : f^*\Omega_Y^{[p]} \rightarrow \Omega_X^{[p]} \quad \text{and} \quad d_{\text{refl}}f : H^0(Y, \Omega_Y^{[p]}) \rightarrow H^0(X, \Omega_X^{[p]}),$$

*which agree with the usual pull-back of Kähler differentials wherever this makes sense.  $\square$*

Theorems 1.1 and its corollary, Theorem 1.2, allow to study reflexive differentials in the context of the minimal model program. These results have been applied to a variety of settings, including a study of hyperbolicity of moduli spaces [KK10]<sup>1</sup>, a partial generalisation of the Beauville–Bogomolov decomposition theorem [GKP11] and deformations of Calabi–Yau varieties [Kol12].

<sup>1</sup>see [Keb11] for a survey

*Main result of this paper.* The main result of this paper asserts the existence of a useful pull-back morphism in a more general setting.

**Theorem 1.3** (Existence of pull-back morphisms in general, compare Theorem 5.2). *Let  $f : X \rightarrow Y$  be any morphism between normal complex varieties. Assume that there exists a Weil divisor  $D$  on  $Y$  such that the pair  $(Y, D)$  is klt. Then there exists a pull-back morphism*

$$d_{\text{refl}} f : f^* \Omega_Y^{[p]} \rightarrow \Omega_X^{[p]},$$

*uniquely determined by natural universal properties.*

*Remark 1.4* (Reference to precise statement). The “natural universal properties” mentioned in Theorem 1.3 are a little awkward to formulate. Precise statements are given in Theorem 5.2 and Section 5.3. In essence, it is required that the pull-back morphisms agree with the pull-back of Kähler differentials wherever this makes sense, and that they satisfy the composition law.

*Discussion of the main result.* It should be noted that Theorem 1.3 does not require the image of  $f$  to intersect the smooth locus of  $Y_{\text{reg}}$ . One particularly relevant setting to which Theorem 1.3 applies is that of a klt space  $Y$ , and the inclusion (or normalisation) of the singular locus, say  $f : X = Y_{\text{sing}} \rightarrow Y$ . It might seem surprising that a pull-back morphism exists in this context, because reflexive differential forms on  $Y$  are, by definition, differential forms defined on the complement of  $Y_{\text{sing}}$ , and no boundary conditions are imposed that would govern the behaviour of those forms near the singularities.

In essence, Theorem 1.3 asserts that differential forms defined on the complement of  $Y_{\text{sing}}$  determine forms on  $Y_{\text{sing}}$ . The following immediate corollary gives a precise formulation.

**Corollary 1.5** (Extension across the singularity stratification). *Let  $Y$  be a normal complex variety with klt singularities. Define a stratification of  $Y$  by disjoint, locally closed, smooth subvarieties,  $(Y^i)_{0 \leq i \leq k} \subseteq Y$ , as follows. Consider the chain of closed subvarieties,*

$$Y = \hat{Y}^0 \supsetneq \hat{Y}^1 \supsetneq \hat{Y}^2 \supsetneq \cdots \supsetneq \hat{Y}^k = \emptyset, \quad \text{where } \hat{Y}^{i+1} := \hat{Y}_{\text{sing}}^i.$$

*Given any index  $i$ , let  $Y^i$  be the smooth locus of  $\hat{Y}^i$ , that is,  $Y^i := \hat{Y}_{\text{reg}}^i$ . Then, given any differential form*

$$\sigma_0 \in H^0(Y^0, \Omega_{Y^0}^p) = H^0(Y_{\text{reg}}, \Omega_{Y_{\text{reg}}}^p),$$

*we obtain a sequence of induced forms  $\sigma_i \in H^0(Y^i, \Omega_{Y^i}^p)$ , for all  $0 < i < k$ .* □

*Remark 1.6.* The singularity stratification defined in Corollary 1.5 is the coarsest stratification whose strata are locally closed and smooth. The conclusion also holds for finer stratifications, such as Whitney stratifications used in the discussion of intersection homology and perverse sheaves.

**1.2. Optimality of the result.** The following two examples show that it is generally not possible to construct reasonable pull-back morphisms for reflexive differentials on even the simplest log canonical spaces.

*Setting 1.7* (Setting for Examples 1.8 and 1.9). Let  $C \subset \mathbb{P}^2$  be a smooth elliptic curve, and let  $X \subset \mathbb{C}^3$  be the affine cone over  $E$ . An elementary computation shows that  $X$  is normal and has an isolated, Gorenstein, log canonical singularity at the vertex point  $x \in X$ . The canonical bundle  $\omega_X = \Omega_X^{[2]}$  is in fact even trivial.

We denote the blow-up of the vertex point  $x$  by  $\beta : \tilde{X} \rightarrow X$ . The variety  $\tilde{X}$  is then smooth. Denoting the  $\beta$ -exceptional curve by  $E$ , there exists an isomorphism

$\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}(-E)$ . Observing that  $\tilde{X}$  is isomorphic to the total space of the line bundle  $\mathcal{O}_C(1)$ , we have constructed the following commutative diagram of morphisms between normal varieties,

$$(1.7.1) \quad \begin{array}{ccccc} E & \xrightarrow{\iota_E, \text{inclusion}} & \tilde{X} & \xrightarrow{\pi, \mathbb{A}^1\text{-bundle}} & C \\ \downarrow c, \text{constant} & & \downarrow \beta, \text{blow-up} & & \\ \{x\} & \xrightarrow{\iota_x, \text{inclusion}} & X & & \end{array}$$

*Example 1.8* (Problems arising from differential forms with poles). We maintain assumptions and notation of Setting 1.7. To give a pull-back map which agrees outside of  $E$  with the usual pull-back map for Kähler differentials, it is then equivalent to give a sheaf morphism

$$\mathcal{O}_{\tilde{X}} \cong \pi^* \omega_X \xrightarrow{d_{\text{refl}} \beta} \omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}(-E)$$

which is isomorphic away from  $E$ . Such a morphism does not exist. If  $\sigma \in H^0(X, \omega_X)$  denotes a global generator of the canonical sheaf, there is no section  $\tilde{\sigma} \in H^0(\tilde{X}, \omega_{\tilde{X}})$  which agrees with  $\sigma$  away from the exceptional set  $E$ .

*Example 1.9* (Impossibility to satisfy composition law). We maintain assumptions and notation of Setting 1.7. We consider reflexive one-forms on  $X$  and assume there were pull-back morphisms

$$d_{\text{refl}} \beta : H^0(X, \Omega_X^{[1]}) \rightarrow H^0(\tilde{X}, \Omega_{\tilde{X}}^{[1]}) \quad \text{and} \quad d_{\text{refl}} \iota_x : H^0(X, \Omega_X^{[1]}) \rightarrow \underbrace{H^0(\{x\}, \Omega_{\{x\}}^{[1]})}_{=\{0\}}$$

satisfying the following natural compatibility conditions.

(1.9.1) Away from the singular point, the map  $d_{\text{refl}} \beta$  agrees with the usual pull-back map of Kähler differentials.

(1.9.2) The composition law holds. In particular, Diagram (1.7.1) will imply that

$$d_{\text{Kähler}} \iota_E \circ d_{\text{refl}} \beta = d_{\text{Kähler}} c \circ d_{\text{refl}} \iota_x$$

where  $d_{\text{Kähler}}$  denotes the usual pull-back of Kähler differentials. Since  $c$  is the constant map, this implies that  $d_{\text{Kähler}} \iota_E \circ d_{\text{refl}} \beta = 0$ .

Let  $\tau_C \in H^0(C, \Omega_C^1) \setminus \{0\}$  be any non-vanishing differential form on the elliptic curve  $C$ , and let  $\tilde{\tau} := d_{\text{Kähler}} \pi(\tau_C) \in H^0(\tilde{X}, \Omega_{\tilde{X}}^1)$  be its pull-back to  $\tilde{X}$ . Since  $\tilde{X} \setminus E$  and  $X_{\text{reg}} = X \setminus \{x\}$  are isomorphic, the form  $\tilde{\tau}$  induces a (reflexive) differential form on  $X$ , say

$$\tau \in H^0(X, \Omega_X^{[1]}) = H^0(X_{\text{reg}}, \Omega_{X_{\text{reg}}}^1).$$

Property (1.9.1) then implies that  $d_{\text{refl}} \beta(\tau) = \tilde{\tau}$ , and

$$(d_{\text{Kähler}} \iota_E \circ d_{\text{refl}} \beta)(\tau) = d_{\text{Kähler}} \iota_E(\tilde{\tau}) \stackrel{(1.9.2)}{=} d_{\text{Kähler}} (\underbrace{\pi \circ \iota_E}_{\text{isomorphism}})(\tau_C) \neq 0.$$

This clearly contradicts Property (1.9.2), showing that pull-back morphisms satisfying these compatibility conditions cannot exist.

In the setting of Example 1.8, there does exist a differential form with logarithmic poles along  $E$ , say  $\tilde{\sigma} \in H^0(\tilde{X}, \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(E))$ , which agrees with  $\sigma$  away from the singular set. One could argue that  $\tilde{\sigma}$  should be taken as a pull-back of  $\sigma$ . While this might be a viable definition when discussing the blow-up morphism  $\beta$ , problems occur as soon as one wishes to pull-back  $\sigma$  via the composition  $\iota_E \circ \beta$ .

In addition to the problems originating from the existence of poles, Example 1.9 shows that there are other and more fundamental reasons why reasonable pull-back maps cannot be defined for log canonical varieties: there is in general no way to define a pull-back map in a way that is compatible with the usual composition law.

**1.3. Idea of proof and outline of paper.** The proof of our main result is technically somewhat involved. The main idea, however, is quite elementary and straightforward. Consider the following simple setting.

*Setting 1.10* (Setting for Examples 1.11 and 1.12). Let  $Y$  be a variety with klt singularities. Assume that the singular locus is a smooth curve  $C$  and that the singularities of  $Y$  are resolved by a single blow-up of the curve  $C$ , say  $\pi : \tilde{Y} \rightarrow Y$ . Assume further that the exceptional set  $E$  of this resolution map is irreducible and smooth over  $C$ .

Let  $\sigma \in H^0(Y, \Omega_Y^{[1]})$  be any given reflexive differential form on  $Y$ . Theorem 1.1 asserts the existence of a regular differential form  $\tilde{\sigma} \in H^0(\tilde{Y}, \Omega_{\tilde{Y}}^1)$  which agrees outside of  $E$  with the form  $\sigma$ .

*Example 1.11* (Pulling back reflexive differentials via a resolution map). In Setting 1.10, define  $\tilde{\sigma}$  as the pull-back of the form  $\sigma$  to  $\tilde{Y}$ . This choice is unique if we require our pull-back form to agree on the smooth locus with the usual pull-back of Kähler differentials.

*Example 1.12* (Pulling back reflexive differentials in more generality). In Setting 1.10, let  $X$  be a smooth variety and  $f : X \rightarrow Y$  a morphism whose image is contained in  $C$ . We aim to define a pull-back form  $\sigma_X \in H^0(X, \Omega_X^1)$ .

A fundamental theorem of Hacon and McKernan, [HM07, Cor. 1.5], asserts that the fibres of  $\pi|_E : E \rightarrow C$  are rationally connected manifolds. Recalling that rationally connected manifolds do not admit non-trivial differential forms, the long exact sequence of the relative differential forms,

$$0 \rightarrow H^0(C, \Omega_C^1) \xrightarrow{d(\pi|_E)} H^0(E, \Omega_E^1) \rightarrow \underbrace{H^0(E, \Omega_{E/C}^1)}_{=0} \rightarrow \cdots,$$

shows that the restriction of  $\tilde{\sigma}$  to  $E$  really is the pull-back of a form  $\tau \in H^0(C, \Omega_C^1)$  on  $C$ . Let  $\sigma_X$  be the standard pull-back of the Kähler differential  $\tau$  to  $X$ .

The choice of  $\sigma_X$  is unique if we require that the pull-back morphisms satisfy the composition law. To this end, recall Graber–Harris–Starr’s generalisation of Tsen’s theorem, [GHS03, Thm. 1.1], which gives the existence of a section  $s : C \rightarrow E \subset \tilde{E}$ , forming a commutative diagram

$$\begin{array}{ccccc} & & E & \xrightarrow{\iota_E, \text{inclusion}} & \tilde{Y} \\ & & \uparrow s & & \downarrow \pi, \text{resolution} \\ X & \xrightarrow{f_C} & C & \xrightarrow{\iota_C, \text{inclusion}} & Y \\ & \searrow f & & & \end{array}$$

Choosing one such  $s$ , observe that

$$\begin{aligned} \sigma_X &= (d_{\text{Kähler}} f_C)(d_{\text{Kähler}} s)(d_{\text{Kähler}} \iota_E)(d_{\text{Kähler}} \pi)(\sigma) \\ &= (d_{\text{Kähler}} f_C)(d_{\text{Kähler}} s)(d_{\text{Kähler}} \iota_E)(\tilde{\sigma}) && \text{by Example 1.11} \\ &= (d_{\text{Kähler}} f_C)(d_{\text{Kähler}} s)(\tilde{\sigma}|_E) \\ &= (d_{\text{Kähler}} f_C)(\tau). \end{aligned}$$

Note that this completely dictates the choice of  $\sigma_X$ .

Two problems occur when trying to adapt the ideas outlined in Examples 1.11–1.12 to the general setting, where  $X$  is allowed to have arbitrary klt singularities.

**Problem 1.13.** *The exceptional set of a resolution morphism need not be irreducible, or relatively smooth over the singular locus. Fibres of resolution maps are known to be rationally chain connected, but Kähler differentials might well exist on these non-normal spaces.*

**Problem 1.14.** *The result of Graber–Harris–Starr is specific to families over 1-dimensional base varieties, and is not known to hold in higher dimensions.*

To overcome the Problem 1.13, we need to consider the sheaves  $\check{\Omega}^p$  of “Kähler differentials modulo torsion” and discuss their properties on reduced, reducible, and not necessarily normal schemes. This is done in Part I of this paper. There, we establish a number of fundamental universal properties and show that reduced, reducible, rationally chain connected schemes with simple normal crossings do not admit any “Kähler differential modulo torsion”. The notions of torsion and torsion-free sheaves on reducible spaces do not seem to be discussed much in the literature. For the reader’s convenience, we recall the definition and establish basic properties in Appendix A.

The Problem 1.14 does not pose fundamental difficulties. However, it does make the proof of our main theorem, given in Part II of this paper, somewhat awkward and lengthy as we constantly need to switch between the spaces in question and suitable coverings, for which a section  $s$  exists.

**1.4. Notation and global conventions.** Throughout the text, we work over the complex number field. This paper discusses several notions of differential forms on singular spaces, and in each case defines pull-back morphisms. To avoid the obvious potential for confusion, we clearly distinguish between the various notions of pulling back.

*Notation 1.15* (Sheaves of differentials and pull-back morphisms). If  $f : X \rightarrow Y$  is any morphism between varieties, we denote the sheaves of Kähler differentials by  $\Omega_X^p$  and  $\Omega_Y^p$ , respectively. The standard pull-back maps of Kähler differentials are denoted as

$$d_{\text{Kähler},f} : f^* \Omega_Y^p \rightarrow \Omega_X^p \quad \text{and} \quad d_{\text{Kähler},f} : H^0(Y, \Omega_Y^p) \rightarrow H^0(X, \Omega_X^p).$$

Part I of this paper discusses the sheaf of Kähler differentials modulo torsion, traditionally denoted as  $\check{\Omega}_X^p := \Omega_X^p / \text{tor}$ . As we will see in Section 2, there exists a meaningful notion of pulling back sections of this sheaf. The associated morphisms of sheaves and vector spaces are denoted as

$$d_{\text{tfree},f} : f^* \check{\Omega}_Y^p \rightarrow \check{\Omega}_X^p \quad \text{and} \quad d_{\text{tfree},f} : H^0(Y, \check{\Omega}_Y^p) \rightarrow H^0(X, \check{\Omega}_X^p).$$

Part II of this paper discusses reflexive differentials. The sheaf of reflexive differentials will always be denoted by  $\Omega_X^{[p]} := (\Omega_X^p)^{**}$ . Following the notation introduced in Theorem 1.2, pull-back morphisms for reflexive differentials will be denoted by  $d_{\text{refl},f}$ .

*Notation 1.16* (Sheaves defined on subschemes). Let  $X$  be a scheme and  $Y \subseteq X$  a subscheme, with associated inclusion map  $\iota : Y \rightarrow X$ . Sheaves  $\mathcal{F}$  on  $Y$  will often be viewed as sheaves on the ambient space  $X$ . If no confusion is likely to arise, we follow standard notation and write  $\mathcal{F}$  as a shorthand for the technically more correct  $\iota_* \mathcal{F}$ .

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## Part I. Torsion-free differentials

### 2. TORSION-FREE DIFFERENTIALS AND THEIR PULL-BACK PROPERTIES

**2.1. The definition of torsion-free differentials.** As indicated in the introduction, we need to discuss sheaves of “Kähler differentials modulo torsion”. The following notation will be used. We refer to Appendix A for a discussion of torsion sheaves on reduced, but possibly reducible spaces.

*Notation 2.1* (Torsion differentials and torsion-free differentials). Let  $X \rightarrow Y$  be a morphism of reduced, quasi-projective schemes. Given any number  $p$ , define  $\check{\Omega}_{X/Y}^p$  as the cokernel of the sequence

$$(2.1.1) \quad 0 \longrightarrow \text{tor } \Omega_{X/Y}^p \xrightarrow{\alpha_{X/Y}} \Omega_{X/Y}^p \xrightarrow{\beta_{X/Y}} \check{\Omega}_{X/Y}^p \longrightarrow 0$$

If  $Y$  is a point, we follow the usual notation and write  $\check{\Omega}_X^p$  instead of  $\check{\Omega}_{X/Y}^p$ . Consider the torsion subsheaf  $\text{tor } \Omega_{X/Y}^p \subseteq \Omega_{X/Y}^p$ , as introduced in Definition A.3 on page 30. Sections in  $\text{tor } \Omega_{X/Y}^p$  and  $\text{tor } \Omega_X^p$  are called (relative) *torsion differentials*. By slight abuse of language, we refer to sections in  $\check{\Omega}_{X/Y}^p$  and  $\check{\Omega}_X^p$  as (relative) *torsion-free differentials*.

*Remark 2.2* (Characterisation of torsion and torsion-free differentials). Torsion differentials are characterised among Kähler differentials as those which vanish at general points of all irreducible components of  $X$ . A torsion-free differential on  $X$  vanishes if and only if it vanishes generically on all irreducible components of  $X$ . We refer to Explanation A.4 for further discussion.

*Notation 2.3* (Morphisms  $\alpha_{X/Y}$  and  $\beta_{X/Y}$ ). Sequence (2.1.1) is obviously of great importance in the discussion of torsion-free differentials. We will therefore maintain the meaning of the symbols  $\alpha_{X/Y}$  and  $\beta_{X/Y}$  throughout the present Section 2. Again, if  $Y$  is a point, we write  $\alpha_X$  instead of  $\alpha_{X/Y}$ .

**2.2. Pull-back properties.** Given a morphism between two varieties, we aim to show that the usual pull-back map of Kähler differentials always induces a pull-back map of torsion-free differentials, even if the image of the morphism is contained in the singular set of the target variety. The following proposition, which asserts that the pull-back of a torsion-differential is always torsion, is a first step in this direction.

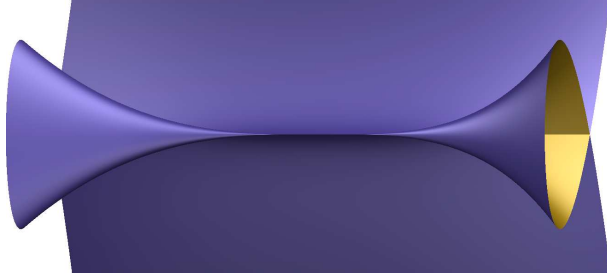
**Proposition 2.4** (Pull-back of torsion differentials are torsion). *Let  $f : X \rightarrow Y$  be a morphism of reduced, quasi-projective schemes. If  $\sigma \in H^0(Y, \text{tor } \Omega_Y^p)$  is a torsion-form on  $Y$ , then  $d_{\text{Kähler}}f(\sigma)$  is a torsion form on  $X$ . In other words,*

$$d_{\text{Kähler}}f(\sigma) \in H^0(X, \text{tor } \Omega_X^p) \subset H^0(X, \Omega_X^p).$$

*Remark 2.5.* The assertion of Proposition 2.4 is clearly true if  $X$  is irreducible and  $f(X)$  intersects the smooth locus of  $Y$ . We claim that Proposition 2.4 also holds in cases where  $f(X)$  is entirely contained in the singular locus. One particularly relevant case is the inclusion of the singular locus, say  $X := Y_{\text{sing}} \hookrightarrow Y$ . If  $Y_{\text{sing}}$  is itself



$$X = \{(x, y, z) \in \mathbb{C}^3 : z^3 + y^2 - x^2 z^2 = 0\}$$



The figure sketches a special case of Proposition 2.4. Here, it can be shown by elementary computation that the restriction of any torsion differential on  $X$  to the singular locus  $X_{\text{sing}} = \{z = y = 0\}$  vanishes. To be more precise, if  $\iota : X_{\text{sing}} \rightarrow X$  denotes the inclusion map, then  $(d_{\text{Kähler}} \iota)|_{\text{tor } \Omega_X^p} = 0$ .

FIGURE 2.1. Restriction of torsion differentials to the singular locus

smooth, then Proposition 2.4 asserts that the pull-back of any torsion differential vanishes. A simple case of this configuration is shown in Figure 2.1.

*Warning 2.6* (Proposition 2.4 is wrong in the relative setup). The proof of Proposition 2.4 relies on the existence of a desingularization map for which no analogue exists in the relative setting. As a matter of fact, Proposition 2.4 becomes wrong when working with relative differentials, unless one makes rather strong additional assumptions. For a simple example, consider a sequence of morphisms,

$$X \xrightarrow[\text{inclusion of except. curve}]{f} Y \xrightarrow[\text{blow-up of smooth surface}]{\pi} Z.$$

A simple computation shows that  $\Omega_{Y/Z}^1 = \text{tor } \Omega_{Y/Z}^1 = f_* \Omega_X^1$ . In particular, if  $U \subset Y$  is any open set and  $\sigma \in H^0(U, \Omega_{Y/Z}^1)$  any non-vanishing differential on  $U$ , then  $\sigma$  is torsion on  $U$ , but  $d_{\text{Kähler}} f(\sigma)$  is a torsion-free differential on the curve  $f^{-1}(U)$ .

Before giving a proof of Proposition 2.4 in Section 2.3, we note a number of corollaries that will be relevant in the further discussion. Among these, the existence of a pull-back map for torsion-free differentials is the most important.

**Corollary 2.7** (Pull-back for sheaves of torsion-free differentials). *Let  $f : X \rightarrow Y$  be a morphism of reduced, quasi-projective schemes. Then there exist unique morphisms  $d_{\text{tor}} f$  and  $d_{\text{tfree}} f$  such that the following diagram with exact rows becomes commutative*

$$(2.7.1) \quad \begin{array}{ccccccc} f^* \text{tor } \Omega_Y^p & \xrightarrow{f^* \alpha_Y} & f^* \Omega_Y^p & \xrightarrow{f^* \beta_Y} & f^* \check{\Omega}_Y^p & \longrightarrow & 0 \\ d_{\text{tor}} f \downarrow & & d_{\text{Kähler}} f \downarrow & & d_{\text{tfree}} \downarrow & & \\ 0 \longrightarrow & \text{tor } \Omega_X^p & \xrightarrow{\alpha_X} & \Omega_X^p & \xrightarrow{\beta_X} & \check{\Omega}_X^p & \longrightarrow 0. \end{array}$$

*Proof.* Proposition 2.4 immediately implies that the restricted morphism  $(d_{\text{Kähler}} f)|_{f^* \text{tor } \Omega_Y^p}$  factorises via the torsion subsheaf  $\text{tor } \Omega_X^p$ ; this is clearly the unique morphism  $d_{\text{tor}} f$  which makes the left square of (2.7.1) commutative. Existence and uniqueness of  $d_{\text{tfree}} f$  follows from surjectivity of  $f^* \beta_Y$  once we note



that

$$\beta_X \circ (d_{\text{Kähler}}f) \circ f^* \alpha_Y = 0.$$

This finishes the proof of Corollary 2.7.  $\square$

The morphism  $f^* \beta_Y$  in (2.7.1) is surjective. This has a number of immediate consequences. We mention two that will be relevant in the further discussion.

**Lemma 2.8.** *In the setting of Corollary 2.7, if  $x \in X$  is any point such where  $d_{\text{Kähler}}f$  vanishes, then  $d_{\text{tfree}}f$  vanishes as well. In other words,*

$$(d_{\text{Kähler}}f)|_{\{x\}} = 0 \Rightarrow (d_{\text{tfree}}f)|_{\{x\}} = 0. \quad \square$$

**Lemma 2.9** (Composition law for pull-back of sheaves of torsion-free differentials). *In the setting of Corollary 2.7, let  $g : X' \rightarrow X$  be any morphism from a reduced, quasi-projective scheme  $X'$ . Then*

$$(2.9.1) \quad d_{\text{tfree}}(f \circ g) = (d_{\text{tfree}}g) \circ g^*(d_{\text{tfree}}f).$$

*Proof.* Consider the commutative diagram of sheaf morphisms,

$$(2.9.2) \quad \begin{array}{ccccc} g^* f^* \Omega_Y^p & \xrightarrow{g^*(d_{\text{Kähler}}f)} & g^* \Omega_X^p & \xrightarrow{d_{\text{Kähler}}g} & \Omega_{X'}^p \\ g^* f^* \beta_Y \downarrow & & \downarrow g^* \beta_X & & \downarrow \beta_{X'} \\ g^* f^* \check{\Omega}_Y^p & \xrightarrow{g^*(d_{\text{tfree}}f)} & g^* \check{\Omega}_X^p & \xrightarrow{d_{\text{tfree}}g} & \check{\Omega}_{X'}^p \end{array}$$

obtained by composing the right commutative square of Diagram (2.7.1) for the morphism  $g$  with the  $g$ -pull-back of the commutative square for the morphism  $f$ . Using the composition law for the pull-back of Kähler differentials,

$$d_{\text{Kähler}}(f \circ g) = (d_{\text{Kähler}}g) \circ g^*(d_{\text{Kähler}}f),$$

the outer square of (2.9.2) is thus written as

$$(2.9.3) \quad \begin{array}{ccc} g^* f^* \Omega_Y^p & \xrightarrow{d_{\text{Kähler}}(f \circ g)} & \Omega_{X'}^p \\ g^* f^* \beta_Y \downarrow & & \downarrow \beta_{X'} \\ g^* f^* \check{\Omega}_Y^p & \xrightarrow{(d_{\text{tfree}}g) \circ g^*(d_{\text{tfree}}f)} & \check{\Omega}_{X'}^p \end{array}$$

As the pull-back of a surjective sheaf morphism, the map  $g^* f^* \beta_Y$  is surjective itself. A comparison of Diagram (2.9.3) with the right square of Diagram (2.7.1) for the composed morphism  $f \circ g$  thus immediately shows Equation (2.9.1), as claimed.  $\square$

*Notation 2.10* (Pull-back for globally defined torsion-free differentials). In the setting of Corollary 2.7, the sheaf morphism  $d_{\text{tfree}}f$  induces a morphism between vector spaces of globally defined torsion-free forms, which we will again denote by  $d_{\text{tfree}}f : H^0(Y, \check{\Omega}_Y^p) \rightarrow H^0(X, \check{\Omega}_X^p)$ .

*Remark 2.11* (Composition law for globally defined torsion-free differentials). In the setting of Corollary 2.7, given a further morphism  $g : X' \rightarrow X$ , Lemma 2.9 implies that the following diagram is commutative,

$$\begin{array}{ccccc} & & d_{\text{tfree}}(f \circ g) & & \\ & \nearrow & & \searrow & \\ H^0(Y, \check{\Omega}_Y^p) & \xrightarrow{d_{\text{tfree}}f} & H^0(X, \check{\Omega}_X^p) & \xrightarrow{d_{\text{tfree}}g} & H^0(X', \check{\Omega}_{X'}^p). \end{array}$$

**2.3. Proof of Proposition 2.4.** We are grateful to Fritz Hörmann for explaining the main idea of the following proof to us. We maintain notation and assumptions of Proposition 2.4 and assume that  $\sigma$  is a torsion form on  $Y$ .

**2.3.1. Simplifications.** Recall from Remark 2.2 or Explanation A.4 that to show that  $d_{\text{Kähler}}f(\sigma)$  is a torsion differential, it suffices to show that it vanishes at general points of each irreducible component of  $X$ . This simple observation has two important consequences.

First consequence: considering one component of  $X$  at a time, we are free to assume without loss of generality that  $X$  is irreducible. Replacing  $X$  by a suitable dense open subset, we can even assume the following.

*Assumption without loss of generality 2.12.* The scheme  $X$  is irreducible and smooth.

Second consequence: Let  $Y_0 \subseteq Y$  be an irreducible component that contains the image  $f(X)$ . The morphism  $f$  factors via the inclusion map,

$$\begin{array}{ccccc} & & f & & \\ X & \xrightarrow{\quad f_0 \quad} & Y_0 & \xrightarrow{\quad \iota_0 \quad} & Y, \end{array}$$

and  $d_{\text{Kähler}}f(\sigma) = (d_{\text{Kähler}}f_0)(d_{\text{Kähler}}\iota_0(\sigma))$ . Since  $\sigma$  is torsion,  $d_{\text{Kähler}}\iota_0(\sigma)$  vanishes at general points of  $Y_0$ , and is therefore a torsion form on  $Y_0$ . Proposition 2.4 will therefore follow for our given morphism  $f$  if we can prove it for  $f_0$ . We can therefore assume without loss of generality that the following holds

*Assumption without loss of generality 2.13.* The scheme  $Y$  is irreducible.

**2.3.2. End of proof.** Let  $\pi_Y : \tilde{Y} \rightarrow Y$  be a resolution of singularities which exists because  $Y$  is irreducible. Choose a component of  $\tilde{X} \subseteq X \times_Y \tilde{Y}$  that surjects onto  $X$ , and let  $\tilde{X}$  be a desingularisation of that component. We obtain a commutative diagram,

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \pi_X \downarrow & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y. \end{array}$$

Since  $\sigma$  vanishes at general points of  $Y$  and since  $\pi_Y$  is birational, it is clear that the pull-back  $d_{\text{Kähler}}\pi_Y(\sigma)$  vanishes at general points of  $\tilde{Y}$ . But since  $\tilde{Y}$  is smooth by Assumption 2.12, that means that it vanishes everywhere,

$$(2.14.1) \quad d_{\text{Kähler}}\pi_Y(\sigma) = 0 \in H^0(\tilde{Y}, \Omega_{\tilde{Y}}^p).$$

Since  $\Omega_X$  is torsion-free and  $\pi_X$  is generically smooth, we see that the pull-back morphism  $d_{\text{Kähler}}\pi_X$  of globally defined forms is in fact injective. It follows that  $d_{\text{Kähler}}f(\sigma) = 0$  if and only if  $(d_{\text{Kähler}}\pi_X)(d_{\text{Kähler}}f(\sigma)) = 0$ . This last pull-back is easily computed to be zero,

$$(d_{\text{Kähler}}\pi_X)(d_{\text{Kähler}}f(\sigma)) = (d_{\text{Kähler}}\tilde{f}) \underbrace{(d_{\text{Kähler}}\pi_Y(\sigma))}_{=0 \text{ by (2.14.1)}} = 0.$$

This ends the proof of Proposition 2.4. □

### 3. TORSION-FREE DIFFERENTIALS IN THE RELATIVE SNC SETTING

In the course of the proof of our main theorem, we will frequently need to consider klt spaces  $X$ , strong resolution maps  $\pi : \tilde{X} \rightarrow X$ , and discuss torsion-free differentials on the exceptional set  $E$ , which is an snc divisor embedded into the smooth space  $\tilde{X}$ . This section gathers several elementary facts about the sheaf  $\check{\Omega}_E^p$  that are needed in the discussion. Most of the material here will be known to experts. We have nonetheless chosen to include full proofs for lack of an adequate reference.

#### 3.1. Relatively snc divisors and associated differentials.

**3.1.1. SNC divisors.** To fix notation used later in this paper, we recall the definition and basic properties of snc pairs.

**Definition 3.1** (SNC pairs [KM98, 0.4(8)]). *Let  $X$  be a normal, quasi-projective variety and  $D$  an effective Weil divisor on  $X$ . Given a point  $x \in X$ , we say that the pair  $(X, D)$  is snc at  $x$  if there exists a Zariski-open neighbourhood  $U$  of  $x$  such that  $U$  is smooth and such that  $\text{supp}(D) \cap U$  is either empty, or a divisor with simple normal crossings. The pair  $(X, D)$  is called snc if it is snc at every point of  $X$ .*

*Given a pair  $(X, D)$ , let  $(X, D)_{\text{reg}}$  be the maximal open set of  $X$  where  $(X, D)$  is snc, and let  $(X, D)_{\text{sing}}$  be its complement, with the induced reduced subscheme structure.*

The following notation and remark can be used to give an alternate definition of “snc pair”. This will be used later to define snc in the relative setting.

**Notation 3.2** (Intersection of boundary components). Let  $(X, D)$  be a pair, where the boundary divisor  $D$  is written as a sum of its irreducible components  $D = \alpha_1 D_1 + \dots + \alpha_n D_n$ . If  $I \subseteq \{1, \dots, n\}$  is any non-empty subset, we consider the scheme-theoretic intersection  $D_I := \bigcap_{i \in I} \text{supp } D_i$ . If  $I$  is empty, set  $D_I := X$ .

**Remark 3.3** (Description of snc pairs). In the setup of Notation 3.2, it is clear that the pair  $(X, D)$  is snc if and only if all  $D_I$  are smooth and of codimension equal to the number of defining equations,  $\text{codim}_X D_I = |I|$  for all  $I$  where  $D_I \neq \emptyset$ .

**3.1.2. SNC morphisms.** The notion of relatively snc divisors has been used in the literature, but its definition has not been discussed much. For the reader’s convenience, we reproduce the definition given in [GKKP11, Sect. 2.B].

**Definition 3.4** (SNC morphism, relatively snc divisor, [VZ02, Def. 2.1]). *If  $(X, D)$  is an snc pair and  $\phi : X \rightarrow T$  a surjective morphism to a smooth variety, we say that  $D$  is relatively snc, or that  $\phi$  is an snc morphism of the pair  $(X, D)$  if for any set  $I$  with  $D_I \neq \emptyset$  all restricted morphisms  $\phi|_{D_I} : D_I \rightarrow T$  are smooth of relative dimension  $\dim X - \dim T - |I|$ .*

**Remark 3.5** (Geometric description of snc morphisms). In the setting of Definition 3.4, assume that  $\phi : X \rightarrow T$  is an snc morphism of the pair  $(X, D)$ . Then, since smooth morphisms are open, each of the sets  $D_I$  dominates  $X$ .

If  $t \in T$  is any point, set  $X_t := \phi^{-1}(t)$  and  $D_t := D \cap X_t$ . Then  $X_t$  is smooth and  $(X_t, D_t)$  is an snc pair.

**Example 3.6** (SNC morphisms). The morphism  $\psi_\alpha$ , shown in Figure 6.1 on page 23 is a prototypical example of an snc morphism. If  $(X, D)$  is an snc pair, then the identity map  $\text{Id}_X : X \rightarrow X$  is an snc morphism if and only if  $D = 0$ . Again assuming that  $(X, D)$  is an snc pair, the constant map from  $X$  to a point is always an snc morphism.

Assume we are given an pair  $(X, D)$  and a point  $x \in X$  such that  $(X, D)$  is snc at  $x$ . It is well-known that there exists a neighbourhood  $U = U(x)$ , open in the analytic topology, and holomorphic coordinates  $x_1, \dots, x_n \in \mathcal{O}_X(U)$  such that  $\text{supp } D \cap U = \{x_1 \cdots x_r = 0\}$ , for a suitable number  $0 \leq r \leq n$ . The following is the relative analogue of this fact.

**Lemma and Definition 3.7** (Adapted coordinates for an snc morphism). *Let  $(X, D)$  be an snc pair and  $\phi : X \rightarrow T$  an snc morphism of the pair  $(X, D)$ . If  $x \in X$  is any point, then there exist neighbourhoods  $V = V(\phi(x)) \subseteq T$  and  $U = U(x) \subseteq \phi^{-1}(V) \subseteq X$  open in the analytic topology, and holomorphic coordinates  $x_1, \dots, x_n \in \mathcal{O}_X(U)$ ,  $y_1, \dots, y_m \in \mathcal{O}_Y(V)$ , centred about  $x$  respectively  $\phi(x)$ , and a number  $0 \leq r \leq n - m$  such that the following holds.*

(3.7.1) *We have  $x_i = y_i \circ \phi$  for all indices  $1 \leq i \leq m$ , and*

(3.7.2) *the support of  $D$  is given as  $\text{supp } D \cap U = \{x_{m+1} \cdots x_{m+r} = 0\}$ .*

*In the setting above, we call the coordinates  $x_\bullet$  and  $y_\bullet$  adapted coordinates for the snc morphism  $\phi$ .  $\square$*

**3.2. Characterisation of torsion-free differentials.** Given an snc pair  $(X, D)$ , the following two propositions describe the sheaf of torsion-free differentials on  $D$  by relating them to the sheaves of differentials on each component of  $D$  and to logarithmic differentials on  $X$ , respectively. These descriptions will later be used in the discussion of relative torsion-free differentials.

**Proposition 3.8** (Torsion-free differentials on snc divisors, I). *Let  $(X, D)$  be an snc pair where  $D$  is reduced, with irreducible components  $D = \cup_i D_i$ . Let  $\phi : X \rightarrow Y$  be a morphism such that  $D$  is relatively snc over  $Y$ . Given any number  $p$ , consider the inclusion maps  $\iota_i : D_i \rightarrow D$  and the natural morphisms*

$$\psi_i : \Omega_{D/Y}^p \rightarrow (\iota_i)_* \Omega_{D_i/Y}^p \quad \text{and} \quad \psi = \oplus \psi_i : \Omega_{D/Y}^p \rightarrow \bigoplus_i (\iota_i)_* \Omega_{D_i/Y}^p.$$

*Then the image of  $\psi$  is naturally isomorphic to the sheaf of torsion-free differentials on  $D$ , that is,  $\text{Image}(\psi) \cong \check{\Omega}_{D/Y}^p$ .*

*Proof.* Recall from Proposition A.8 that the push-forward sheaves  $(\iota_i)_* \Omega_{D_i/Y}^p$  are torsion-free sheaves on the reducible space  $D$ . Since subsheaves of torsion-free sheaves are torsion-free, Corollary A.7, it follows that  $\text{Image}(\psi)$  is itself torsion-free. The universal property of torsion-freeness, Proposition A.6, therefore gives a factorisation of  $\psi$  as follows:

$$\Omega_{D/Y}^p \xrightarrow{\quad \psi \quad} \check{\Omega}_{D/Y}^p \xrightarrow{\quad \check{\psi} \quad} \text{Image}(\psi)^\subset \longrightarrow \bigoplus_i (\iota_i)_* \Omega_{D_i/Y}^p.$$

To finish the proof, it suffices to show that  $\check{\psi}$  is injective. That, however, follows from Proposition A.9 because  $\check{\psi}$  is generically injective: as a matter of fact, both  $\psi$  and  $\check{\psi}$  are isomorphic away from the singular set of  $D$ .  $\square$

In the absolute case, the following statement appears without proof in [Nam01b, p. 129].

**Proposition 3.9** (Torsion-free differentials on snc divisors, II). *In the setup of Proposition 3.8, there exists a short exact sequence<sup>2</sup>*

$$(3.9.1) \quad 0 \longrightarrow \Omega_{X/Y}^p(\log D) \otimes \mathcal{J}_D \xrightarrow{A_{X/Y}^p} \Omega_{X/Y}^p \xrightarrow{B_{X/Y}^p} \check{\Omega}_{D/Y}^p \longrightarrow 0.$$

<sup>2</sup>The sheaf  $\check{\Omega}_{D/Y}^p$  of Sequence (3.9.1) should be seen as a sheaf on  $X$ , cf. Notation 1.16 on page 6.

*Proof.* Viewing  $\Omega_{D/Y}^p$  and  $\Omega_{D_i/Y}^p$  as sheaves on  $X$ , Proposition 3.8 asserts that to prove Proposition 3.9, it suffices to show that  $\Omega_{X/Y}^p(\log D) \otimes \mathcal{I}_D$  is exactly the kernel of the composed map

$$\Omega_{X/Y}^p \xrightarrow{r} \Omega_{D/Y}^p \xrightarrow{\psi} \bigoplus_i \Omega_{D_i/Y}^p.$$

$\eta$  (curved arrow from  $\Omega_{X/Y}^p$  to  $\bigoplus_i \Omega_{D_i/Y}^p$ )

Setting  $\eta_i := \psi_i \circ r : \Omega_{X/Y}^p \rightarrow \Omega_{D_i/Y}^p$ , the kernel of  $\eta$  is then this intersection of the kernels of all the  $\eta_i$ , that is,

$$(3.9.2) \quad \ker \eta = \bigcap_i \ker \eta_i \subset \Omega_{X/Y}^p.$$

The kernels of the  $\eta_i$  are well understood. As a matter of fact, an elementary computation in adapted coordinates shows<sup>3</sup> that

$$(3.9.3) \quad \ker \eta_i = \Omega_{X/Y}^p(\log D_i) \otimes \mathcal{I}_{D_i}.$$

In particular, we see that  $\ker \eta_i$  are locally free subsheaves of  $\Omega_{X/Y}^p$ . Since the intersection of two reflexive subsheaves is reflexive, Equations (3.9.2) and (3.9.3) show that  $\ker \eta$  is a reflexive sheaf on  $X$ , and isomorphic to  $\Omega_{X/Y}^p(\log D) \otimes \mathcal{I}_D$  on the open set  $X \setminus D_{\text{sing}}$ . Since  $\text{codim}_X D_{\text{sing}} \geq 2$ , and since two reflexive sheaves agree if and only if they agree on the complement of a small set, we obtain that  $\ker \eta \cong \Omega_{X/Y}^p(\log D) \otimes \mathcal{I}_D$  on all of  $X$ , thus finishing the proof of Proposition 3.9.  $\square$

Given a smooth morphism  $X \rightarrow Y$ , it is well-known that the restriction of  $\Omega_{X/Y}^p$  to any fibre  $F$  equals  $\Omega_F^p$ . Using the descriptions given above, we show that the same holds for torsion-free differentials in the relative snc setting.

**Corollary 3.10** (Restriction to fibres). *In the setup of Proposition 3.8, let  $y \in Y$  be a smooth point of the image  $\phi(D)$ , and consider the preimage  $D_y := (\phi|_D)^{-1}(y)$ . Then*

$$(3.10.1) \quad \check{\Omega}_{D/Y}^p|_{D_y} \cong \check{\Omega}_{D_y}^p.$$

*Proof.* Set  $X_y := \phi^{-1}(y)$  and  $D_y := D \cap X_y$ , and recall from Remark 3.5 that  $(X_y, D_y)$  is an snc pair. We observe that to prove Corollary 3.10, it suffices to show Equation (3.10.1) in the neighbourhood of any given point  $x \in D_y$ . Given one such  $x$ , choose open neighbourhoods  $U = U(x)$  and  $V = V(y)$  and adapted coordinates  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$  as in Lemma and Definition 3.7. Using these coordinates, an elementary computation shows that the restriction of Sequence (3.9.1) to the fibre  $X_y$  stays exact, and that the terms of the restricted sequence are identified as follows,

$$(3.10.2) \quad 0 \longrightarrow \underbrace{(\Omega_{X/Y}^p(\log D) \otimes \mathcal{I}_D)|_{X_y}}_{=\Omega_{X_y}^p(\log D_y) \otimes \mathcal{I}_{D_y}} \longrightarrow \underbrace{\Omega_{X/Y}^p|_{X_y}}_{=\Omega_{X_y}^p} \longrightarrow \check{\Omega}_{D/Y}^p|_{X_y} \longrightarrow 0.$$

Recall from Example 3.6 that the constant map from  $X_y$  to a point is an snc morphism of the pair  $(X_y, D_y)$ . An application of Proposition 3.9 therefore identifies the cokernel of Sequence (3.10.2) as  $\check{\Omega}_{D_y}^p$ , finishing the proof.  $\square$

<sup>3</sup>In the absolute case, this description of  $\ker \eta_i$  is also found in [EV92, Prop. 2.3(c) on p. 13].

**3.3. Filtrations for torsion-free differentials in the relative snc setting.** Given a smooth morphism  $X \rightarrow Y$ , the sequence of relative differentials on  $X$  induces a canonical filtration on the sheaf  $\Omega_X^p$ . The following proposition shows that the same statement holds for torsion-free differentials in the relative snc setting.

**Proposition 3.11** (Filtration of relative torsion-free differentials for an snc morphism). *Let  $(X, E)$  be an snc pair, where  $E$  is a reduced divisor on  $X$ . Let  $\phi : X \rightarrow Y$  a morphism such that  $E$  is relatively snc over  $Y$ . If  $0 \leq p \leq \dim X - 1$  is any number, then there exists a filtration*

$$\check{\Omega}_E^p = \check{\mathcal{F}}^0 \supseteq \check{\mathcal{F}}^1 \supseteq \dots \supseteq \check{\mathcal{F}}^p \supseteq \check{\mathcal{F}}^{p+1} = 0$$

and for all  $0 \leq r \leq p$  sequences

$$(3.11.1) \quad 0 \rightarrow \check{\mathcal{F}}^{r+1} \rightarrow \check{\mathcal{F}}^r \rightarrow (\phi|_E)^* \Omega_Y^r \otimes \check{\Omega}_{E/Y}^{p-r} \rightarrow 0.$$

*Proof.* Since  $\phi$  is smooth, the sequence of relative differentials on  $X$  is a short exact sequence of locally free sheaves on  $X$ ,

$$0 \rightarrow \phi^* \Omega_Y^1 \rightarrow \Omega_X^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0.$$

Following [Har77, II.5 Ex 5.16], there exists an induced filtration

$$\Omega_X^p = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \dots \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} = 0$$

and sequences for all  $0 \leq r \leq p$ ,

$$0 \rightarrow \mathcal{F}^{r+1} \rightarrow \mathcal{F}^r \rightarrow \phi^* \Omega_Y^r \otimes \Omega_{X/Y}^{p-r} \rightarrow 0.$$

Recalling Sequence (3.9.1) of Proposition 3.9, we define filtrations of  $\Omega_{X/Y}^p(\log E) \otimes \mathcal{J}_E$  and  $\check{\Omega}_{E/Y}^p$  by setting

$$\begin{aligned} \mathcal{F}^r(\log) &:= (A_{X/Y}^p)^{-1}(\mathcal{F}^r) && \dots \text{ filtration of } \Omega_{X/Y}^p(\log E) \otimes \mathcal{J}_E, \text{ and} \\ \check{\mathcal{F}}^r &:= B_{X/Y}^p(\mathcal{F}^r) && \dots \text{ filtration of } \check{\Omega}_{E/Y}^p. \end{aligned}$$

With these definitions, it is clear that  $\check{\mathcal{F}}^r \cong \mathcal{F}^r / \mathcal{F}^r(\log)$ , for all indices  $r$ . We aim to understand the sheaves  $\check{\mathcal{F}}^r$  in more detail. An explicit computation in adapted coordinates, which we leave to the reader, reveals two facts. First, there is an isomorphism

$$\mathcal{F}^r(\log) / \mathcal{F}^{r+1}(\log) \cong \phi^* \Omega_Y^r \otimes \Omega_{X/Y}^{p-r}(\log E) \otimes \mathcal{J}_E.$$

Secondly, one obtains the description of the natural map between quotients given in the following commutative diagram

$$(3.11.2) \quad \begin{array}{ccc} \mathcal{F}^r(\log) / \mathcal{F}^{r+1}(\log) & \xrightarrow[\text{natl. map between quotients}]{q} & \mathcal{F}^r / \mathcal{F}^{r+1} \\ \cong \downarrow & & \downarrow \cong \\ \phi^* \Omega_Y^r \otimes (\Omega_{X/Y}^{p-r}(\log E) \otimes \mathcal{J}_E) & \xrightarrow{\text{Id}_{\phi^* \Omega_Y^r} \otimes A_{X/Y}^{p-r}} & \phi^* \Omega_Y^r \otimes \Omega_{X/Y}^{p-r}. \end{array}$$

In particular, since  $A_{X/Y}^{p-r}$  is injective and since  $\phi^* \Omega_Y^r$  is locally free on  $X$ , it follows that  $q$  is injective. The Snake Lemma thus yields a commutative diagram of

coherent sheaves on  $X$  as follows,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{F}^{r+1}(\log) & \longrightarrow & \mathcal{F}^r(\log) & \longrightarrow & \mathcal{F}^r(\log) / \mathcal{F}^{r+1}(\log) \longrightarrow 0 \\
 & & \downarrow A_{X/Y}^p|_{\mathcal{F}^{r+1}(\log)} & & \downarrow A_{X/Y}^p|_{\mathcal{F}^r(\log)} & & \downarrow q \\
 (3.11.3) \quad 0 & \longrightarrow & \mathcal{F}^{r+1} & \longrightarrow & \mathcal{F}^r & \longrightarrow & \mathcal{F}^r / \mathcal{F}^{r+1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \check{\mathcal{F}}^{r+1} & \longrightarrow & \check{\mathcal{F}}^r & \longrightarrow & \check{\mathcal{F}}^r / \check{\mathcal{F}}^{r+1} \longrightarrow 0.
 \end{array}$$

In summary, we obtain the following description of the successive quotients in the filtration of  $\check{\Omega}_E^p$ ,

$$\begin{aligned}
 \check{\mathcal{F}}^r / \check{\mathcal{F}}^{r+1} &\cong \mathcal{F}^r / \mathcal{F}^{r+1} / \mathcal{F}^r(\log) / \mathcal{F}^{r+1}(\log) && \text{Diag. (3.11.3)} \\
 &\cong \phi^* \Omega_Y^r \otimes \Omega_{X/Y}^{p-r} / q \left( \phi^* \Omega_Y^r \otimes \left( \Omega_{X/Y}^{p-r}(\log E) \otimes \mathcal{I}_E \right) \right) && \text{Diag. (3.11.2)} \\
 &\cong \phi^* \Omega_Y^r \otimes \left( \Omega_{X/Y}^{p-r} / A_{X/Y}^{p-r} \left( \Omega_{X/Y}^{p-r}(\log E) \otimes \mathcal{I}_E \right) \right) && \phi^* \Omega_Y^r \text{ loc. free} \\
 &\cong \phi^* \Omega_Y^r \otimes \check{\Omega}_{E/Y}^{p-r} && \text{Seq (3.9.1)} \\
 &\cong (\phi|_E)^* \Omega_Y^r \otimes \check{\Omega}_{E/Y}^{p-r}.
 \end{aligned}$$

This finishes the proof of Proposition 3.11.  $\square$

#### 4. TORSION-FREE DIFFERENTIALS ON RATIONALLY CHAIN CONNECTED SPACES

It is well-known that rationally chain connected compact manifolds do not admit any differential forms. Here, we show that the same holds for torsion-free differentials on rationally chain connected varieties with arbitrary singularities.

**Theorem 4.1** (Torsion-free differentials on rationally chain connected spaces). *Let  $X$  be a reduced, projective scheme. Assume that  $X$  is rationally chain connected. Then*

$$H^0(X, \check{\Omega}_X^p) = 0, \quad \text{for all } 0 < p \leq \dim X.$$

*Remark 4.2.* In Theorem 4.1, we do not assume that  $X$  is irreducible.

*Warning 4.3* (Kähler differentials on rationally chain connected spaces). The statement of Theorem 4.1 becomes wrong if one replaces torsion-free differentials with Kähler differentials. For an example, let  $X = X_1 \cup X_2$  be the union of two distinct lines in  $\mathbb{P}^2$ . The reducible variety  $X$  is then clearly rationally chain connected. The sheaf  $\Omega_X^1$  of Kähler differentials contains a non-trivial torsion subsheaf, supported at the intersection point  $X_1 \cap X_2$ . As a skyscraper sheaf supported at a single point,  $\text{tor } \Omega_X^1$ , and hence  $\Omega_X^1$ , do have non-trivial sections.

Similar examples exist where  $X$  is irreducible and rationally connected. The paper [GR11] discusses cones where  $\text{tor } \Omega_X^1$  is non-trivial, and supported at the vertex point.

*Warning 4.4* (Reflexive differentials on rationally chain connected spaces). The statement of Theorem 4.1 becomes wrong if one replaces torsion-free differentials with reflexive differentials; Example 1.9 on page 4 discusses a non-trivial reflexive form on the (rationally chain connected) cone over an elliptic curve.

The main reason for failure Theorem 4.1 in this setting is the fact that  $X$  has log canonical rather than klt singularities. For rationally chain connected spaces



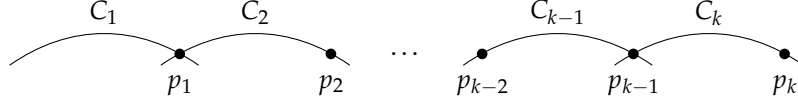


FIGURE 4.1. A pointed chain of smooth rational curves, as used in the proof of Theorem 4.1

$X$  with klt singularities, it is shown in [GKKP11, Thm. 5.1] that  $H^0(X, \Omega_X^{[p]}) = 0$  for all  $p$ . For reflexive tensor operation other than  $\wedge^{[p]}$ , the question becomes rather subtle. We refer to [GKP11, Sect. 3] for a discussion of known facts and for examples.

The arguments used to prove of Theorem 4.1 follow the standard proof for the non-existence of Kähler differentials on rationally connected manifolds.

**4.1. Proof of Theorem 4.1.** Let  $\sigma \in H^0(X, \check{\Omega}_X^p)$  be any reflexive differential. By Remark 2.2, it suffices to show that  $\sigma$  vanishes at the general point of every irreducible component of  $X$ . In other words, given any irreducible component  $X_0 \subseteq X$  with inclusion map  $\iota : X_0 \rightarrow X$ , we need to show that  $d_{\text{tfree}}\iota(\sigma) = 0$ .

The assumption that  $X$  is rationally chain connected implies that there exists a fixed point  $y \in X$ , such that general points of  $X_0$  can be connected to  $y$  using a finite-length chain of rational curves. The following is thus an immediate consequence.

**Consequence 4.5.** *There exists a nodal chain of smooth rational curves  $C = C_1 \cup \dots \cup C_k$  with marked points  $p_i$  as shown in Figure 4.1, a smooth, irreducible, quasi-projective variety  $H \subset \text{Hom}(C, X)$  and a morphism  $\mu : H \times C \rightarrow X$  with the following properties.*

(4.5.1) *The variety  $H \times C_1$  dominates the component  $X_0$ .*

(4.5.2) *The restriction of  $\mu$  to  $H \times \{p_k\}$  is constant.* □

With the notation introduced above, the following Lemma is key to the proof of Theorem 4.1.

**Lemma 4.6.** *Setup as above. For any index  $1 \leq j \leq k$ , we consider the following two restrictions of  $\mu$ ,*

$$\mu_{p_j} : H \times \{p_j\} \rightarrow X \quad \text{and} \quad \mu_{C_j} : H \times C_j \rightarrow X.$$

*Then the following two equations hold for all indices  $1 \leq j \leq k$*

$$(4.6.1) \quad d_{\text{tfree}}\mu_{p_j}(\sigma) = 0 \in H^0(H \times \{p_j\}, \Omega_{H \times \{p_j\}}^p) \quad \text{and}$$

$$(4.6.2) \quad d_{\text{tfree}}\mu_{C_j}(\sigma) = 0 \in H^0(H \times C_j, \Omega_{H \times C_j}^p)$$

Assuming for second that Lemma 4.6 holds, consider Equation (4.6.2) for  $j = 1$ . Write  $\mu_{C_1} : H \times C_1$  as a composition

$$H \times C_1 \xrightarrow{\mu_{C_1,0}} X_0 \xrightarrow{\iota} X$$

Equation (4.6.2) and the composition law for pull-back of torsion-free differentials, Lemma 2.9 give a vanishing of forms,

$$0 = d_{\text{tfree}}\mu_{C_1}(\sigma) = d_{\text{tfree}}\mu_{C_1,0}(d_{\text{tfree}}\iota(\sigma))$$

Recalling from (4.5.1) that  $\mu_{C_1,0}$  is dominant, Equation (4.6.2) says that the restriction of  $\sigma$  to the component  $X_0$  vanishes generically. The required vanishing of  $d_{\text{tfree}}\iota(\sigma)$  then an immediate consequence. Theorem 4.1 is thus a consequence of Lemma 4.6.

4.1.1. *Proof of Lemma 4.6.* It remains to prove Lemma 4.6. The proof proceeds by descending induction on  $j$ . To this end, the following statements will be shown in Section 4.1.2–4.1.4 below.

*Claim 4.7* (Start of induction). Equation (4.6.1) holds for  $j = k$ .

*Claim 4.8* (Inductive step I). For all indices  $1 \leq j \leq k$ , Equation (4.6.2) holds for the index  $j$  if Equation (4.6.1) holds for  $j$ .

*Claim 4.9* (Inductive step II). For all indices  $1 < j \leq k$ , Equation (4.6.1) holds for the index  $j - 1$  if Equation (4.6.2) holds for  $j$ .

4.1.2. *Proof of Claim 4.7.* For  $j = k$ , Equation (4.6.1) follows from (4.5.2), which asserts that  $\mu_{p_k}$  is constant. The pull-back map of Kähler differentials is thus zero,  $d\mu_{p_k} = 0$ . By Lemma 2.8, the pull-back map of torsion-free differentials is therefore zero too, so that  $d_{\text{tfree}}\mu_{p_k} = 0$  as claimed.  $\square$

4.1.3. *Proof of Claim 4.8.* Let  $1 \leq j \leq k$  be any given index, and assume that Equation (4.6.1) holds for  $j$ . The following morphisms are relevant in our discussion

$$\begin{array}{ccccc} & & \mu_{p_j} & & \\ & \nearrow & & \searrow & \\ H \times \{p_j\} & \xrightarrow{\gamma, \text{inclusion}} & H \times C_j & \xrightarrow{\mu_{C_j}} & X \\ & & \downarrow \pi_H, \text{projection} & & \\ & & H & & \end{array}$$

The product structure of  $H \times C_j$  immediately gives a splitting

$$\Omega_{H \times C_j}^p \cong \underbrace{\pi_H^*(\Omega_H^p)}_{=: \mathcal{A}} \oplus \underbrace{\pi_H^*(\Omega_H^{p-1}) \otimes \pi_{C_j}^*(\Omega_{C_j}^1)}_{=: \mathcal{B}}$$

Decompose  $d_{\text{tfree}}\mu_{C_j}(\sigma) \in H^0(H \times C_j, \Omega_{H \times C_j}^p)$  accordingly as  $d_{\text{tfree}}\mu_{C_j}(\sigma) = \check{\sigma}_{\mathcal{A}} + \check{\sigma}_{\mathcal{B}}$ , we aim to show that both  $\check{\sigma}_{\mathcal{A}}$  and  $\check{\sigma}_{\mathcal{B}}$  are zero.

First, if  $F = \{h\} \times C_j \cong \mathbb{P}^1$  is any fibre of  $\pi_H$ , then  $\mathcal{B}|_F \cong \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus \bullet}$  is anti-ample. It follows that  $\mathcal{B}$  only has the trivial section, so  $\check{\sigma}_{\mathcal{B}} = 0$ . Secondly, it follows from Equation (4.6.1) and from the composition law for pull-back of torsion-free differentials, Lemma 2.9, that

$$d_{\text{tfree}}\gamma(d_{\text{tfree}}\mu_{C_j}(\sigma)) = \check{\mu}_{p_j}(\sigma) = 0.$$

We obtain that  $\check{\sigma}_{\mathcal{A}}|_{H \times \{p_j\}} = 0$ . Since  $\mathcal{A}|_F$  is a trivial vector bundle,  $\mathcal{A}|_F \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus \bullet}$ , this shows vanishing of  $\check{\sigma}_{\mathcal{A}}$ .

In summary, we have seen that  $d_{\text{tfree}}\mu_{C_j}(\sigma)$  is zero, as asserted in Claim 4.8.  $\square$

4.1.4. *Proof of Claim 4.9.* Let  $1 < j \leq k$  be any given index, and assume that Equation (4.6.2) holds for the index  $j$ . Since  $p_{j-1} \in C_j$ , we obtain a factorisation

$$\begin{array}{ccccc} & & \mu_{p_{j-1}} & & \\ & \nearrow & & \searrow & \\ H \times \{p_{j-1}\} & \xrightarrow{\gamma, \text{inclusion}} & H \times C_j & \xrightarrow{\mu_{C_j}} & X \end{array}$$

The composition law for the pull-back of torsion-free differentials, Lemma 2.9, then asserts that

$$d_{\text{tfree}}\mu_{p_{j-1}}(\sigma) = d_{\text{tfree}}\gamma(d_{\text{tfree}}\mu_{C_j}(\sigma)) \stackrel{(4.6.2)}{=} d_{\text{tfree}}\gamma(0) = 0,$$

which immediately shows the desired vanishing. This finishes the proof of Claim 4.9, and hence of Theorem 4.1.  $\square$

## Part II. Pull-back properties of reflexive differentials on klt spaces

### 5. MAIN RESULT AND ELEMENTARY CONSEQUENCES

**5.1. Formulation of the main result.** We aim to construct a pull-back map for reflexive differentials, which will turn out to be uniquely determined by universal properties. To formulate these properties (“composition law”, “compatibility with Kähler differentials”) in a technically correct manner, it seems easiest to use the language of functors. The following definition fixes the category.

**Definition 5.1** (Category of klt base spaces). *Let  $X$  be a normal, irreducible variety. We call  $X$  a klt base space if there exists a  $\mathbb{Q}$ -divisor  $D$  on  $X$  such that the pair  $(X, D)$  is klt. A morphism between klt base spaces is simply a morphism of varieties.*

The following theorem contains the main result of this paper. Its proof is given in Sections 6–7, starting on Page 21. An extension of Theorem 5.2 to morphisms with arbitrary domain is discussed in Section 5.3 on page 21.

**Theorem 5.2** (Pull-back map for reflexive differentials on klt base spaces). *There exists a unique contravariant functor*

$$(5.2.1) \quad \begin{array}{ccc} d_{\text{refl}} : \{ \text{klt base spaces} \} & \rightarrow & \{ \mathbb{C}\text{-vector spaces} \} \\ X & \mapsto & H^0(X, \Omega_X^{[p]}) \end{array}$$

that satisfies the following “Compatibility with Kähler differentials”. If  $f : Z \rightarrow X$  is a morphism of klt base spaces such that the open set  $Z^\circ := Z_{\text{reg}} \cap f^{-1}(X_{\text{reg}})$  is not empty, then there exists a commutative diagram,

$$(5.2.2) \quad \begin{array}{ccc} H^0(X, \Omega_X^{[p]}) & \xrightarrow{d_{\text{refl}}f} & H^0(Z, \Omega_Z^{[p]}) \\ \text{restriction}_X \downarrow & & \downarrow \text{restriction}_Z \\ H^0(X_{\text{reg}}, \Omega_{X_{\text{reg}}}^p) & \xrightarrow{d_{\text{Kähler}}(f|_{Z^\circ})} & H^0(Z^\circ, \Omega_{Z^\circ}^p), \end{array}$$

where  $d_{\text{Kähler}}(f|_{Z^\circ})$  denotes the usual pull-back of Kähler differentials, and  $d_{\text{refl}}f$  denotes the linear map of complex vector spaces induced by the contravariant functor (5.2.1).

**Remark 5.3** (Restrictions used in Diagram (5.2.2)). In the setting discussed in Diagram (5.2.2), we have equalities

$$\Omega_X^{[p]}|_{X_{\text{reg}}} = \Omega_{X_{\text{reg}}}^p \quad \text{and} \quad \Omega_Z^{[p]}|_{Z^\circ} = \Omega_{Z^\circ}^p.$$

This justifies the use of the word “restriction” in Diagram (5.2.2). Since  $X$  and  $Z$  are normal, hence smooth in codimension one, the restriction maps  $\text{restriction}_X$  and  $\text{restriction}_Z$  are clearly isomorphic.

**Remark 5.4** (Compatibility with Kähler differentials in special cases). Consider a morphism  $f : Z \rightarrow X$  of klt base spaces whose image is contained in the singular locus of  $X$ . In this case, the set  $Z^\circ$  discussed in Theorem 5.2 is empty. The compatibility condition formulated in the theorem is then also empty, that is, always satisfied. This does not mean that  $d_{\text{refl}}f$  is an arbitrary map. The pull-back map  $d_{\text{refl}}f$  is in fact uniquely defined by the functorial properties (“composition rule”), and by the requirement that the pull-back maps of other, dominant, morphisms need to satisfy compatibility with Kähler differentials. Later in this paper, the proof of Proposition 5.9 will illustrate this principle.

**Remark 5.5** (Sanity of notation). Given a klt base space, the functorial formulation of Theorem 5.2 would in principle allow to write  $d_{\text{refl}}X$  as a shorthand for the space

$H^0(X, \Omega_X^{[p]})$  of reflexive differential forms on  $X$ . In order to avoid confusion and incompatibility with the literature, we will never use this notation.

**5.2. Elementary properties of the pull-back map.** The compatibility with Kähler differentials implies that practically all properties known from the pull-back map of Kähler differentials also hold in the reflexive setting. We mention some of the more immediate examples in the present Section 5.2.

**Proposition 5.6** (Compatibility with open immersions). *Let  $X$  be any klt base space,  $Z \subseteq X$  any open set and  $f : Z \rightarrow X$  the inclusion map. Then*

$$d_{\text{refl}}f : H^0(X, \Omega_X^{[p]}) \rightarrow H^0(Z, \Omega_Z^{[p]})$$

*equals the standard restriction map.*

*Proof.* Define the subset  $Z^\circ$  as in Theorem 5.2 and observe that  $Z^\circ$  is not empty. Using the isomorphisms  $\text{restriction}_X$  and  $\text{restriction}_Y$ , the claim follows from the observation that  $d_{\text{Kähler}}(f|_{Z^\circ})$  is the standard restriction map.  $\square$

**Proposition 5.7** (Morphisms to smooth target spaces). *Let  $f : Z \rightarrow X$  be any dominant morphism between klt base spaces, where  $X$  is smooth. Then  $d_{\text{refl}}f$  equals the composition of the following maps*

$$H^0(X, \Omega_X^{[p]}) = H^0(X, \Omega_X^p) \xrightarrow{d_{\text{Kähler}}} H^0(Z, \Omega_Z^p) \xrightarrow{\varphi} H^0(Z, \Omega_Z^{[p]}),$$

*where  $\varphi$  is induced by the standard map from the sheaf  $\Omega_Z^p$  into its double dual.*  $\square$

**Proposition 5.8** (Pull-back via a dominant morphism). *Let  $f : Z \rightarrow X$  be any dominant morphism between klt base spaces. Then the associated map*

$$d_{\text{refl}}f : H^0(X, \Omega_X^{[p]}) \rightarrow H^0(Z, \Omega_Z^{[p]})$$

*is injective.*

*Proof.* Define the subset  $Z^\circ$  as in Theorem 5.2 and observe that  $Z^\circ$  is not empty, and that  $f|_{Z^\circ} : Z^\circ \rightarrow X$  is dominant. Proposition 5.8 then follows immediately from injectivity of  $d_{\text{Kähler}}(f|_{Z^\circ})$ .  $\square$

**5.2.1. The pull-back map on sheaf level.** The following Proposition 5.9 and Corollary 5.10 imply that the pull-back map for reflexive differentials is already defined on sheaf level, just as the pull-back map for Kähler differentials is.

**Proposition 5.9** (Compatibility with module multiplication). *Let  $f : Z \rightarrow X$  be any morphism between klt base spaces. Then associated map*

$$d_{\text{refl}}f : H^0(X, \Omega_X^{[p]}) \rightarrow H^0(Z, \Omega_Z^{[p]})$$

*is a morphism of  $\mathcal{O}_X(X)$ -modules.*

**Corollary 5.10** (Pull-back map on sheaf level). *If  $f : Z \rightarrow X$  is any morphism between klt base spaces, then the pull-back morphism*

$$d_{\text{refl}}f : H^0(X, \Omega_X^{[p]}) \rightarrow H^0(Z, \Omega_Z^{[p]})$$

*is induced by a sheaf morphism*

$$d_{\text{refl}}f : f^*\Omega_X^{[p]} \rightarrow \Omega_Z^{[p]}.$$

*Proof of Corollary 5.10.* Immediate from compatibility with open immersions and compatibility with module multiplication, Propositions 5.6 and 5.9.  $\square$

The proof of Proposition 5.9 makes use of the following elementary construction, which we note for later reference.

*Construction 5.11.* Let  $f : Z \rightarrow X$  be any morphism between normal varieties, and let  $\pi : \tilde{X} \rightarrow X$  be a resolution of singularities. Then there exists a smooth variety  $V$  and a commutative diagram,

$$(5.11.1) \quad \begin{array}{ccccc} & & & \tilde{X} & \\ & & a \curvearrowright & \downarrow \pi & \text{resolution of} \\ & & & & \text{singularities} \\ V & \xrightarrow{g, \text{ surjective}} & Z & \xrightarrow{f} & X. \end{array}$$

One way of construction goes as follows. Choose a component  $Y \subseteq \tilde{X} \times_X Z$  that surjects onto  $Z$ , and let  $\tilde{Y} \rightarrow Y$  be a desingularization. Let  $d$  be the relative dimension of  $\tilde{Y} \rightarrow Z$ , and  $V \subset \tilde{Y}$  the intersection of  $d$  general hyperplanes (if  $d = 0$ , set  $V := \tilde{Y}$ ). The variety  $V$  is then smooth, with natural morphisms to  $Z$  and  $\tilde{X}$  making the diagram commutative. The morphism to  $g : V \rightarrow Z$  constructed in this manner is surjective and generically finite.

*Proof of Proposition 5.9.* Given reflexive forms  $\sigma_1, \sigma_2 \in H^0(X, \Omega_X^{[p]})$  and a function  $\tau \in H^0(X, \Omega_X^{[p]})$ , we need to show that

$$(5.12.1) \quad (d_{\text{refl}} f)(\tau \cdot \sigma_1 + \sigma_2) \stackrel{!}{=} \tau \cdot (d_{\text{refl}} f)(\sigma_1) + (d_{\text{refl}} f)(\sigma_2).$$

There are two situation where this is easily true.

- (5.12.2) If the target of  $f$  is smooth, then Proposition 5.7 implies that (5.12.1) holds because both  $d_{\text{Kähler}}$  and  $\varphi$  are  $\mathcal{O}_X(X)$ -linear.
- (5.12.3) If  $f$  is dominant, then  $Z^\circ$  is non-empty, and Equation (5.12.1) holds because compatibility with Kähler differentials implies that it holds on the open set  $Z^\circ$ .

Proposition 5.9 is hence shown for dominant morphisms, and for morphisms with smooth target. If  $f$  is neither, apply Construction 5.11 to obtain a commutative diagram as in (5.11.1). We have seen above that  $d_{\text{refl}} g$  is  $\mathcal{O}_Z(Z)$ -linear. It is injective by Proposition 5.8. To prove (5.12.1), it will therefore suffice to prove the analogous equation for the composed morphism  $f \circ g$ ,

$$(d_{\text{refl}}(f \circ g))(\tau \cdot \sigma_1 + \sigma_2) \stackrel{!}{=} \tau \cdot (d_{\text{refl}}(f \circ g))(\sigma_1) + (d_{\text{refl}}(f \circ g))(\sigma_2).$$

That, however, follows from functoriality,  $d_{\text{refl}}(f \circ g) = (d_{\text{refl}} \pi) \circ (d_{\text{refl}} a)$ , because we have seen in above that  $d_{\text{refl}} \pi$  and  $d_{\text{refl}} a$  are linear over global sections of their respective structure sheaves.  $\square$

**5.2.2. Compatibility with wedge products and exterior derivatives.** If  $X$  is any normal variety, then to give a reflexive differential on  $X$  it is equivalent to give a Kähler differential on the smooth locus  $X_{\text{reg}}$ . More precisely, if  $\iota : X_{\text{reg}} \rightarrow X$  denotes the inclusion of the smooth locus into  $X$ , then  $\Omega_X^{[p]} = \iota_* \Omega_{X_{\text{reg}}}^p$ . This description of reflexive differentials immediately allows to define *reflexive wedge products*, that is, for all  $p$  and  $q$  morphisms

$$\begin{aligned} \wedge : \quad & \Omega_X^{[p]} \otimes_{\mathcal{O}_X} \Omega_X^{[q]} \rightarrow \Omega_X^{[p+q]} \\ \wedge : \quad & H^0(X, \Omega_X^{[p]}) \otimes_{\mathcal{O}_X(X)} H^0(X, \Omega_X^{[q]}) \rightarrow H^0(X, \Omega_X^{[p+q]}) \end{aligned}$$

which, on the smooth locus of  $X$ , agree with the usual wedge products. We define  $\mathbb{C}$ -linear *reflexive exterior derivatives* in the same fashion, for all  $p$ ,

$$\begin{aligned} d : \quad & \Omega_X^{[p]} \rightarrow \Omega_X^{[p+1]} \\ d : \quad & H^0(X, \Omega_X^{[p]}) \rightarrow H^0(X, \Omega_X^{[p+1]}). \end{aligned}$$

In this setting, the proof of Proposition 5.9 applies almost verbatim to give the following compatibility result.

**Proposition 5.13** (Compatibility with wedge products and exterior derivatives). *Let  $f : Z \rightarrow X$  be any morphism between normal varieties. Then the pull-back maps for reflexive differentials and sheaves of reflexive differentials commute with reflexive wedge products and reflexive exterior derivatives.*  $\square$

**5.3. Morphisms with arbitrary domain.** The properties listed above allow for the construction of a meaningful pull-back map even for morphisms where only the target is assumed to be a klt base space.

To this end, let  $f : Z \rightarrow X$  be any morphism between normal varieties and assume that  $X$  a klt base space. Observing that the smooth locus  $Z_{\text{reg}}$  is a klt base space as well, and recalling that the restriction map

$$\text{restriction}_Z : H^0(Z, \Omega_Z^{[p]}) \rightarrow H^0(Z_{\text{reg}}, \Omega_{Z_{\text{reg}}}^p)$$

is isomorphic, define a pull-back map for reflexive differentials as the composition of the following two maps,

$$H^0(X, \Omega_X^{[p]}) \xrightarrow{d_{\text{refl}}(f|_{X_{\text{reg}}})} H^0(Z_{\text{reg}}, \Omega_{Z_{\text{reg}}}^p) \xrightarrow{\text{restriction}_Z^{-1}} H^0(Z, \Omega_Z^{[p]}).$$

As there is no possibility of confusion, we denote this map again by  $d_{\text{refl}}f$ . It is easy to show that all properties listed in Section 5.2 also hold for this generalised map. In particular,  $d_{\text{refl}}f$  is induced by a sheaf morphism. We have thus constructed maps  $d_{\text{refl}}f : \Omega_X^{[p]} \rightarrow \Omega_Z^{[p]}$  and  $d_{\text{refl}}f : H^0(X, \Omega_X^{[p]}) \rightarrow H^0(Z, \Omega_Z^{[p]})$  for all indices  $p$ .

## 6. PREPARATION FOR THE PROOF OF THEOREM 5.2

Given a morphism of klt base spaces and a reflexive form on the target space, the following proposition constructs a reflexive form on the domain which satisfies a weak universal property.

**Proposition 6.1** (Construction of pull-back forms). *Let  $f : Z \rightarrow X$  be any morphism of klt base spaces, and let  $\sigma \in H^0(X, \Omega_X^{[p]})$  be any reflexive  $p$ -form on  $X$ . Then there exists a unique reflexive  $p$ -form  $\tau \in H^0(Z, \Omega_Z^{[p]})$  satisfying the following universal property. Given any commutative diagram*

$$(6.1.1) \quad \begin{array}{ccc} & & \tilde{X} \\ & \nearrow a & \downarrow \pi \text{ resolution of singularities} \\ V & \xrightarrow{g, \text{ surjective}} Z_{\text{reg}} \xrightarrow{f|_{Z_{\text{reg}}}} & X, \end{array}$$

where  $V$  is smooth, let  $\tilde{\sigma} \in H^0(\tilde{X}, \Omega_{\tilde{X}}^p)$  be the unique differential form on  $\tilde{X}$  which agrees with  $\sigma$  wherever  $\pi$  is isomorphic. Then

$$(6.1.2) \quad d_{\text{Kähler}}a(\tilde{\sigma}) = d_{\text{Kähler}}g(\tau|_{Z_{\text{reg}}}) \in H^0(V, \Omega_V^p).$$

**Remark 6.2** (Existence of  $\tilde{\sigma}$ , restriction of reflexive differentials). The existence of the form  $\tilde{\sigma}$  used in the formulation of Proposition 6.1 is the main result of the paper [GKKP11]. We refer to [GKKP11, Thm. 1.4 and Rem. 1.5.2] for details. In analogy with Remark 5.3, we have used the canonical identification  $\Omega_Z^{[p]}|_{Z_{\text{reg}}} \cong \Omega_{Z_{\text{reg}}}^p$  implicitly in the formulation of Equation (6.1.2).

*Remark 6.3* (The case where  $f(Z) \not\subset X_{\text{sing}}$ ). In the setting of Proposition 6.1, if  $f(Z)$  is not contained in the singular locus of  $X$ , then it follows immediately from the universal property (6.1.2) that  $\tau$  is the unique reflexive form whose restriction to the open set  $Z^\circ := Z_{\text{reg}} \cap f^{-1}(X_{\text{reg}})$  satisfies  $\tau|_{Z^\circ} = d_{\text{Kähler}}(f|_{Z^\circ})(\sigma|_{X_{\text{reg}}})$ .

*Remark 6.4* (Existence of  $V$ ,  $a$  and  $g$  for given resolution  $\pi$ ). Given an arbitrary resolution of singularities  $\pi : \tilde{X} \rightarrow X$ , Construction 5.11 shows that there always exist smooth varieties  $V$  and morphisms  $a, g$  as in Diagram (6.1.1).

**6.1. Preparation for the proof of Proposition 6.1.** We will see in Section 6.2 that Proposition 6.1 is in fact a corollary of the following, seemingly weaker lemma.

**Lemma 6.5** (Weak version of Proposition 6.1). *Let  $f : Z \rightarrow X$  be any morphism of klt base spaces, and let  $\sigma \in H^0(X, \Omega_X^{[p]})$  be any reflexive  $p$ -form on  $X$ . Then there exists a resolution of singularities  $\pi : \tilde{X} \rightarrow X$  and a reflexive  $p$ -form  $\tau \in H^0(Z, \Omega_Z^{[p]})$  satisfying a universal property similar to the one spelled out in Proposition 6.1: given any smooth variety  $V$  and morphisms  $a, g$  forming a commutative diagram as in (6.1.1), then Equation (6.1.2) holds.*

*Remark 6.6* (Relation between Lemma 6.5 and Proposition 6.1). We will later see that the form  $\tau$  constructed in Lemma 6.5 for one specific resolution map will also work for any other.

The remainder of the present Section 6.1 is devoted to the proof of Lemma 6.5. Though not extremely involved on a conceptual level, the proof is somewhat lengthy to write down. To help the reader maintain an overview, we have subdivided the proof into a number of relatively independent steps, given in Sections 6.1.1–6.1.6 below.

**6.1.1. Proof of Lemma 6.5: setup of notation, simplification.** We maintain assumptions and notation of Lemma 6.5 throughout the proof. In order to construct a reflexive form on  $Z$ , it suffices to construct the form away from a set of codimension two. Since  $Z$  is a klt base space, hence normal, we can therefore assume without loss of generality that the following holds.

**Additional Assumption 6.7.** *The space  $Z$  is smooth.*

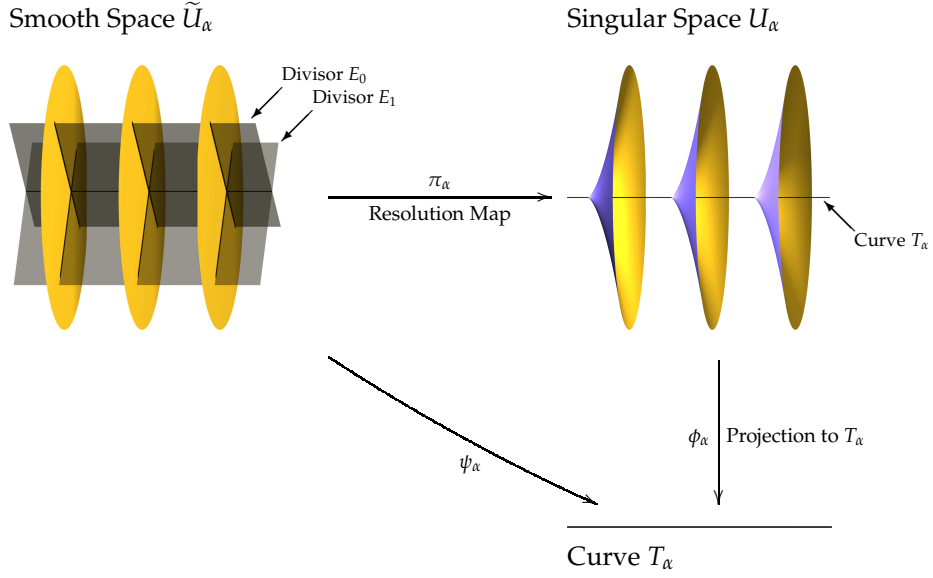
In case where the image of  $f$  is not contained in the singular set of  $X$ , it has been shown in [GKKP11, Thm. 4.3] that there exists a differential form  $\tau$  on  $Z$  which agrees on the open set  $f^{-1}(X_{\text{reg}})$  with the usual pull-back of the Kähler differential  $\sigma|_{X_{\text{reg}}}$ . The differential  $\tau$  clearly satisfies all requirements stated in Lemma 6.5, so that the proof is already finished in this case. We will therefore assume without loss of generality that the following holds.

**Additional Assumption 6.8.** *The image of  $f$  is contained in the singular set of  $X$ .*

We define  $T \subseteq X$  as the Zariski closure of the image of  $f$ , that is,  $T := \overline{f(Z)}$ . Choose a desingularization  $\pi : \tilde{X} \rightarrow X$  with the additional property that the preimage  $\pi^{-1}(T) \subset \tilde{X}$  has pure codimension one and forms a divisor with simple normal crossing support. Finally, let  $E \subset \pi^{-1}(T)$  be the union of those components that dominate (=surject onto) the irreducible variety  $T$ . Its irreducible components are denoted as  $E = E_0 \cup \dots \cup E_k$ .

**6.1.2. Proof of Lemma 6.5: projection to general points of  $T$ .** One way to describe the geometry of  $X$  near general points of  $T$  is by looking at a family of sufficiently general complete intersections  $(H_t)_{t \in T}$ , and by studying the varieties  $H_t$  at their intersection points with  $T$ . At general points of  $T$ , the family defines a morphism,





The figure sketches the situation of Lemma 6.9 in case where  $X$  is a threefold and  $T$  is a curve. Over  $T_\alpha$ , the exceptional set of the resolution map  $\pi$  is a reducible divisor,  $E = E_0 \cup E_1$ . The composed map  $\psi_\alpha$  is an snc morphism of the pair  $(\tilde{U}_\alpha, E_\alpha)$ .

FIGURE 6.1. Projection to general points of  $T$

and it is often notationally convenient to discuss the varieties  $H_i$  as being fibres of that morphism. This idea is not new, and is explained in great detail in [GKKP11, Sect. 2.G]. The following lemma summarises the results and fixes notation used throughout the remainder of the proof of Lemma 6.5. Figure 6.1, taken from the preprint version of the paper [GKKP11], illustrates the setup.

**Lemma and Notation 6.9** (Projection to general points of  $T$ ). *In the setup of Section 6.1.1, there exists a dense, Zariski-open set  $X^\circ \subseteq X$ , an open covering of  $T^\circ := T \cap X^\circ$  by sets  $U_\alpha \subseteq X^\circ$  that are open in the analytic topology, and commutative diagrams of holomorphic morphisms,*

$$\begin{array}{ccccc}
 & \tilde{U}_\alpha & \xrightarrow{\tilde{\iota}_\alpha, \text{inclusion}} & \tilde{X}^\circ \subseteq \tilde{X} & \\
 \psi_\alpha \swarrow & \downarrow \pi|_{\tilde{U}_\alpha} & & \downarrow \pi|_{\tilde{X}^\circ} & \downarrow \pi \\
 T_\alpha & \xleftarrow{\phi_\alpha} U_\alpha & \xrightarrow{\iota_\alpha, \text{inclusion}} & X^\circ \subseteq X & 
 \end{array}$$

where

$$\tilde{X}^\circ := \pi^{-1}(X^\circ), \quad \tilde{U}_\alpha := \pi^{-1}(U_\alpha), \quad T_\alpha := T \cap U_\alpha,$$

such that the following extra conditions hold

- (6.9.1) The variety  $T^\circ$  is smooth and not empty. The sheaf  $\Omega_{T^\circ}^1$  is trivial.
- (6.9.2) Setting  $E^\circ := E \cap \tilde{X}^\circ$ , we have  $(\pi^\circ)^{-1}(T^\circ) = E^\circ$ .
- (6.9.3) If  $E_j^\circ \subseteq E^\circ$  is any irreducible component, then  $\pi^\circ|_{E_j^\circ} : E_j^\circ \rightarrow T^\circ$  is smooth.
- (6.9.4) The restrictions  $\phi_\alpha|_{T_\alpha} : T_\alpha \rightarrow T_\alpha$  are the identity morphisms.
- (6.9.5) The holomorphic maps  $\psi_\alpha$  are smooth. The divisors  $E_\alpha := E \cap \tilde{U}_\alpha$  are relatively snc over  $T_\alpha$ .

*Proof.* The existence of an open set  $X^\circ$  satisfying Conditions (6.9.1)–(6.9.3) is clear. Shrinking  $X^\circ$  further, if necessary, the standard technique of “projection to a subvariety”, explained and proven in [GKKP11, Prop. 2.26], yields the existence of a finite, étale covering map  $\gamma : Y \rightarrow X^\circ$  and a commutative diagram,

$$\begin{array}{ccccc}
 & & \tilde{Y} & \xrightarrow{\tilde{\gamma}, \text{ finite, étale}} & \tilde{X}^\circ \subseteq \tilde{X} \\
 & \searrow \psi & \downarrow p & & \downarrow \pi|_{\tilde{X}^\circ} \\
 B & \xleftarrow{\phi} & Y & \xrightarrow{\gamma, \text{ finite, étale}} & X^\circ \subseteq X, \\
 & & & & \downarrow \pi
 \end{array}$$

where  $\tilde{Y} := \tilde{X}^\circ \times_{X^\circ} Y$ , where  $B := \gamma^{-1}(T^\circ)$ , and where  $p$  and  $\tilde{\gamma}$  denote the obvious projections. The morphism  $\phi$  has the additional property that its restriction to  $B \subset Y$  is the identity map. Shrinking  $X^\circ$  further, if necessary, generic smoothness of morphisms and its logarithmic analogue, [GKKP11, Rem. 2.11], allow to assume that  $\psi$  is smooth, and that the divisor  $E^\circ \times_{X^\circ} Y \subsetneq \tilde{Y}$  is relatively snc over  $B$ .

To end the proof, it suffices to find a covering of  $T^\circ \subset X^\circ$  by analytically open sets  $(U_\alpha)_{\alpha \in A} \subseteq X^\circ$  that are small enough so that their preimages are disjoint unions of  $(\deg \gamma)$ -many open sets,

$$\gamma^{-1}(U_\alpha) = V_{\alpha,1} \dot{\cup} \cdots \dot{\cup} V_{\alpha,\deg \gamma},$$

each canonically identified with  $U_\alpha$ . Choosing one  $V_{\alpha,\bullet}$  for each given set  $U_\alpha$ , the morphisms  $\phi_\alpha$  and  $\psi_\alpha$  are immediately obtained from these identifications. Conditions (6.9.4) and (6.9.5) follow from the properties of  $\phi$  and  $\psi$ .  $\square$

**6.1.3. Proof of Lemma 6.5: construction of a differential form on  $T^\circ$ .** We follow the ideas outlined in Section 1.3. Given any point  $t \in T_\alpha$ , a fundamental result of Hacon-McKernan asserts that the fibre  $E_t := \pi^{-1}(t)$  is rationally chain connected. Using the results obtained in Section 4, this implies that relative differentials in  $\Omega_{U_\alpha/T_\alpha}^p$  vanish modulo torsion when restricted to any component of  $E_t$ . This is a first indication of the principle that “the restriction of any differential to  $E^\circ$  comes from  $T^\circ$ ”, as formulated and proven in the following lemma.

**Lemma 6.10** (Restriction of any differential to  $E^\circ$  comes from  $T^\circ$ ). *In the setup of Section 6.1.2, the pull-back map of torsion-free differentials,*

$$d_{\text{tfree}}(\pi|_{E^\circ}) : H^0(T^\circ, \check{\Omega}_{T^\circ}^p) \rightarrow H^0(E^\circ, \check{\Omega}_{E^\circ}^p),$$

*is isomorphic.*

*Proof.* Choosing an open cover of  $T^\circ$  as in Lemma 6.9, it suffices to show that the pull-back maps associated with the restricted morphisms,

$$d_{\text{tfree}}(\pi|_{E_\alpha}) : H^0(T_\alpha, \check{\Omega}_{T_\alpha}^p) \rightarrow H^0(E_\alpha, \check{\Omega}_{E_\alpha}^p),$$

are isomorphisms, for all  $\alpha \in A$ . Let  $\alpha \in A$  be any given index. The pull-back map  $d_{\text{tfree}}(\pi|_{E_\alpha})$  is clearly injective. To prove surjectivity, consider the filtration and sequences introduced in Proposition 3.11,

$$(6.10.1) \quad \check{\Omega}_{E_\alpha}^p = \check{\mathcal{F}}^0 \supseteq \check{\mathcal{F}}^1 \supseteq \cdots \supseteq \check{\mathcal{F}}^p \supseteq \check{\mathcal{F}}^{p+1} = 0$$

$$(6.10.2) \quad 0 \rightarrow \check{\mathcal{F}}^{r+1} \rightarrow \check{\mathcal{F}}^r \rightarrow (\psi_\alpha^* \check{\Omega}_{T_\alpha}^r) \otimes \check{\Omega}_{E_\alpha/T_\alpha}^{p-r} \rightarrow 0,$$

which exist for all  $0 \leq r \leq p$ . Observe that in case where  $r = p$ , Sequence (6.10.2) yields  $\check{\mathcal{F}}^p = \psi_\alpha^* \check{\Omega}_{T_\alpha}^p$ . To prove Lemma 6.10, we aim to show that

$$(6.10.3) \quad H^0(E_\alpha, \check{\Omega}_{E_\alpha}^p) \stackrel{!}{=} H^0(E_\alpha, \check{\mathcal{F}}^p) = H^0(E_\alpha, \psi_\alpha^* \check{\Omega}_{T_\alpha}^p) = H^0(T_\alpha, \check{\Omega}_{T_\alpha}^p).$$

Since the vector bundles,  $\psi_\alpha^* \check{\Omega}_{T_\alpha}^r = \psi_\alpha^* \Omega_{T_\alpha}^r$  are trivial by Assertion (6.9.1) of Lemma 6.9, Equation (6.10.3) will follow from an inductive argument using the Sequences (6.10.2) once we show that

$$(6.10.4) \quad H^0(E_\alpha, \check{\Omega}_{E_\alpha/T_\alpha}^q) \stackrel{!}{=} 0 \quad \text{for all } q > 0.$$

Using Corollary 3.10 to identify the restriction  $\check{\Omega}_{E_\alpha/T_\alpha}^q|_{E_t}$  with  $\check{\Omega}_{E_t}^q$  for all points  $t \in T_\alpha$  and all fibres  $E_t := \pi^{-1}(t) \subset E_\alpha$ , Equation (6.10.4) will in turn follow from the stronger claim that

$$(6.10.5) \quad H^0(E_t, \check{\Omega}_{E_t}^q) \stackrel{!}{=} 0, \quad \text{for all } t \in T_\alpha \text{ and all } q > 0.$$

To prove (6.10.5), recall that  $X^\circ$  is a klt base space. A fundamental result of Hacon-McKernan [HM07, Cor. 1.5(1)] thus implies that fibres  $E_t$ , being fibres of the birational resolution map  $\pi$ , are rationally chain connected. The vanishing asserted in (6.10.5) is therefore an immediate consequence of the non-existence of torsion-free forms in rationally chain connected spaces, as asserted in Theorem 4.1.  $\square$

*Remark 6.11.* Using the standard fact that klt base spaces have rational singularities, it might be possible to give a proof of Lemma 6.10 using Namikawa's analysis of mixed Hodge structures, [Nam01a, Lemma 1.2], rather than the more elementary Theorem 4.1

**Corollary 6.12** (Restriction of  $\tilde{\sigma}$  to  $E^\circ$  comes from a form  $\eta^\circ$  on  $T^\circ$ ). *In the setup of Section 6.1.2, there exists a unique torsion-free form  $\eta^\circ \in H^0(T^\circ, \check{\Omega}_{T^\circ}^p)$  such that*

$$(6.12.1) \quad d_{\text{tfree}} \iota^\circ(\tilde{\sigma}) = d_{\text{tfree}}(\pi|_{E^\circ})(\eta^\circ),$$

where  $\iota^\circ : E^\circ \rightarrow \tilde{X}^\circ$  denotes the obvious inclusion.  $\square$

6.1.4. *Proof of Lemma 6.5: construction of a differential form on an open set of  $Z$ .* Consider the non-empty, Zariski-open set  $Z^\circ := f^{-1}(T^\circ) \subseteq Z$ . We obtain a torsion-free form

$$(6.12.2) \quad \tau^\circ := (d_{\text{tfree}}(f|_{Z^\circ}))(\eta^\circ) \in H^0(Z^\circ, \check{\Omega}_{Z^\circ}^p).$$

The following elementary lemma summarises what we know about  $\tau^\circ$ . In essence, it shows that the form  $\tau^\circ$  satisfies Equation (6.1.2) on the open set  $Z^\circ$ .

**Lemma 6.13** (The form  $\tau^\circ$  satisfies Equation (6.1.2) on the open set  $Z^\circ$ ). *Setting as above. Given a smooth variety  $V$  and morphisms  $a, g$  forming a commutative diagram as in (6.1.1) of Proposition 6.1,*

$$(6.13.1) \quad \begin{array}{ccccc} & & & \tilde{X} & \\ & & a & \nearrow & \\ V & \xrightarrow{g, \text{surjective}} & Z_{\text{reg}} & \xrightarrow{f|_{Z_{\text{reg}}}} & X, \\ & & & \downarrow \pi & \text{resolution of singularities} \end{array}$$

set  $V^\circ := g^{-1}(Z^\circ)$ . Then

$$(6.13.2) \quad (d_{\text{tfree}}(a|_{V^\circ}))(\tilde{\sigma}) = (d_{\text{tfree}}(g|_{V^\circ}))(\tau^\circ).$$

*Proof.* It follows from commutativity of Diagram (6.13.1) and from Assumption 6.8 that the image of the restricted morphism  $a|_{V^\circ}$  is contained in the divisor  $E^\circ \subset \tilde{X}$ . In other words, there exists a factorisation

$$(6.13.3) \quad \begin{array}{ccccc} & & a|_{V^\circ} & \nearrow & \\ V^\circ & \xrightarrow{b} & E^\circ & \xrightarrow{\iota^\circ, \text{inclusion}} & X^\circ. \end{array}$$

Equation (6.13.2) then follows easily from the composition law for pull-back of torsion-free differentials formulated in Lemma 2.9 and from the commutativity of the diagrams considered so far.

$$\begin{aligned}
(d_{\text{tfree}}(g|_{V^\circ}))(\tau^\circ) &= (d_{\text{tfree}}(g|_{V^\circ}) \circ d_{\text{tfree}}(f|_{Z^\circ}))(\eta^\circ) && \text{Definition of } \tau^\circ \text{ in (6.12.2)} \\
&= (d_{\text{tfree}}b \circ d_{\text{tfree}}(\pi|_{E^\circ}))(\eta^\circ) && \text{Commutativity of (6.1.1)} \\
&= (d_{\text{tfree}}b \circ d_{\text{tfree}}\iota^\circ)(\tilde{\sigma}) && \text{Equation (6.12.1)} \\
&= (d_{\text{tfree}}(a|_{\tilde{V}^\circ}))(\tilde{\sigma}) && \text{Factorisation (6.13.3).}
\end{aligned}$$

This finishes the proof of Lemma 6.13.  $\square$

*Remark 6.14* (Viewing  $\tau^\circ$  as a Kähler differential). The varieties  $V$ ,  $V^\circ$ ,  $Z$  and  $Z^\circ$  are smooth by Assumption 6.7. The spaces  $H^0(Z^\circ, \check{\Omega}_{Z^\circ}^p)$  and  $H^0(Z^\circ, \Omega_{Z^\circ}^p)$  therefore agree, allowing us to view the torsion-free form  $\tau^\circ \in H^0(Z^\circ, \check{\Omega}_{Z^\circ}^p)$  as a Kähler differential, say  $\tau_{\text{Kähler}}^\circ \in H^0(Z^\circ, \Omega_{Z^\circ}^p)$ . Since  $a|_{V^\circ}$  and  $g|_{V^\circ}$  are morphisms between smooth spaces, the pull-back notions for torsion-free forms and Kähler differentials agree. Equation (6.14.1) therefore implies the following equality of Kähler differentials

$$(6.14.1) \quad (d_{\text{Kähler}}(a|_{V^\circ}))(\tilde{\sigma}) = (d_{\text{Kähler}}(g|_{V^\circ}))(\tau_{\text{Kähler}}^\circ).$$

6.1.5. *Proof of Lemma 6.5: construction of a differential form on  $Z$ .* We show that  $\tau_{\text{Kähler}}^\circ$  can be extended from the open set  $Z^\circ$  to all of  $Z$ . The following criterion, essentially shown in [GKK10], will be employed.

**Lemma 6.15** (Regularity criterion for differential forms). *Setup as above. If there exists a smooth, irreducible variety  $\tilde{V}$ , a proper, generically finite surjective morphism  $\tilde{g} : \tilde{V} \rightarrow Z$  and a differential form  $\tilde{\tau} \in H^0(\tilde{V}, \Omega_{\tilde{V}}^p)$  whose restriction to  $\tilde{V}^\circ := \tilde{g}^{-1}(Z^\circ)$  agrees with the pull-back of  $\tau_{\text{Kähler}}^\circ$ ,*

$$(d_{\text{Kähler}}(g|_{V^\circ}))(\tau_{\text{Kähler}}^\circ) = \tilde{\tau}|_{V^\circ},$$

*then there exists a Kähler differential  $\tau_{\text{Kähler}} \in H^0(Z, \Omega_Z^p)$  which extends  $\tau_{\text{Kähler}}^\circ$ .*

*Proof.* Recalling from Assumption 6.7 that  $Z$  is smooth, the sheaf  $\Omega_Z^p$  is locally free, thus reflexive. We are therefore free to shrink  $Z$  by removing a suitable set of codimension two, and assume without loss of generality that  $\tilde{g}$  is actually finite. For finite morphisms between smooth spaces, Lemma 6.15 has already been shown in [GKK10, Cor. 2.12(ii)].  $\square$

**Corollary 6.16** (Regularity of  $\tau$ ). *Setting as above. Then there exists a Kähler differential  $\tau_{\text{Kähler}} \in H^0(Z, \Omega_Z^p)$  which extends  $\tau_{\text{Kähler}}^\circ$ .*

*Proof.* Recall from Remark 6.4 there always exist smooth varieties  $V$  and morphisms  $a, g$  forming a Diagram as in (6.1.1), where  $g$  is generically finite. Setting  $V^\circ := g^{-1}(Z^\circ)$  as before, Equation (6.14.1) of Remark 6.14 asserts that the differential form  $(d_{\text{Kähler}}(g|_{V^\circ}))(\tau_{\text{Kähler}}^\circ)$  initially defined only on  $V^\circ$  extends to give a regular differential form on all of  $V$ , namely  $(d_{\text{Kähler}}(a|_{V^\circ}))(\tilde{\sigma})$ . Lemma 6.15 therefore applies.  $\square$

6.1.6. *Proof of Lemma 6.5: end of proof.* Given a smooth variety  $V$  and morphisms  $a, g$  forming a Diagram as in (6.1.1), we need to show that Equation (6.1.2) holds. We have seen in Lemma 6.13 and Remark 6.14 that the equation holds on  $Z^\circ$ . Since  $Z$  is smooth by Assumption 6.7, the sheaf  $\Omega_Z^p$  is torsion-free. Since  $Z^\circ$  is open in Zariski topology, hence dense, Equation (6.1.2) holds everywhere, as required. This finishes the proof of Lemma 6.5.  $\square$

**6.2. Proof of Proposition 6.1.** To prove Proposition 6.1, assume that we are given a morphism of klt base spaces,  $f : Z \rightarrow X$ , and a reflexive  $p$ -form on  $X$ , say  $\sigma \in H^0(X, \Omega_X^{[p]})$ .

**6.2.1. Proof of Proposition 6.1: Existence of the form  $\tau$ .** The weak version of Proposition 6.1 shown in Lemma 6.5 asserts the existence of a resolution map  $\pi : \tilde{X} \rightarrow X$  and a reflexive  $p$ -form  $\tau \in H^0(Z, \Omega_Z^{[p]})$  satisfying universal property (6.1.2). To prove Proposition 6.1, we will show that the form  $\tau$  satisfies the stronger requirements of Proposition 6.1. To this end, assume we are given an arbitrary resolution of singularities  $\pi' : \tilde{X}' \rightarrow X$  with form  $\tilde{\sigma}' \in H^0(\tilde{X}', \Omega_{\tilde{X}'}^p)$ , a smooth variety  $V'$  and morphisms  $a', g'$  forming a commutative diagram as in (6.1.1). We need to show that the following equation holds,

$$(6.16.1) \quad d_{\text{Kähler}} a'(\tilde{\sigma}') = d_{\text{Kähler}} g'(\tau|_{Z_{\text{reg}}}) \in H^0(V, \Omega_V^p).$$

To this end, choose a component of the fibred product  $\tilde{X}' \times_X \tilde{X}$  that surjects onto  $X$ , and let  $\hat{X}$  be a desingularization of that component. We obtain a resolution of singularities  $\hat{\pi} : \hat{X} \rightarrow X$  that dominates both  $\tilde{X}_1$  and  $\tilde{X}_1$ . In a similar vein, choose a component of  $V' \times_X \hat{X}$  that surjects onto  $V'$ . Desingularising, we obtain a smooth variety  $V$  and a large commutative diagram of morphisms between varieties,

$$\begin{array}{ccccccc} & & & & \hat{X} & \xlongequal{\quad} & \hat{X} \\ & & & & \downarrow p' & & \downarrow p \\ & & & & \tilde{X}' & & \tilde{X} \\ & & & & \downarrow \pi' & & \downarrow \pi \\ & & & & X & \xlongequal{\quad} & X \\ V & \xrightarrow{g'', \text{ surjective}} & V' & \xrightarrow{g', \text{ surjective}} & Z_{\text{reg}} & \xrightarrow{f|_{Z_{\text{reg}}}} & X \end{array}$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image. The image shows a more complex diagram with curved arrows  $a''$  and  $a'$  from  $V$  to  $\hat{X}$  and  $\tilde{X}'$  respectively, and a curved arrow  $\hat{\pi}$  from  $\hat{X}$  to  $\tilde{X}$ .)

Observing that the pull-back of the forms  $\tilde{\sigma}$  and  $\tilde{\sigma}'$  to the common resolution  $\hat{X}$  agree on the preimage of  $X_{\text{reg}}$ , we obtain an equality of differential forms,

$$(6.16.2) \quad d_{\text{Kähler}} p'(\tilde{\sigma}') = d_{\text{Kähler}} p(\tilde{\sigma}).$$

An application of Lemma 6.5 with  $a = p \circ a''$ ,  $g = g' \circ g''$  therefore yields the following chain of equalities,

$$\begin{aligned} (d_{\text{Kähler}} g'' \circ d_{\text{Kähler}} g')(\tau) &= (d_{\text{Kähler}} a'' \circ d_{\text{Kähler}} p)(\tilde{\sigma}) && \text{Lemma 6.5} \\ &= (d_{\text{Kähler}} a'' \circ d_{\text{Kähler}} p')(\tilde{\sigma}') && \text{Equation (6.16.2)} \\ &= (d_{\text{Kähler}} g'' \circ d_{\text{Kähler}} a')(\tilde{\sigma}') && \text{Equality } p' \circ a'' = a' \circ g'' \end{aligned}$$

Equality (6.16.1) now follows because  $g''$  is surjective and  $d_{\text{Kähler}} g''$  therefore injective. We have thus shown that  $\tau$  satisfies the requirements of Proposition 6.1.

**6.2.2. Proof of Proposition 6.1: Uniqueness of the form  $\tau$ .** Remark 6.4 asserts the existence of a diagram as in (6.1.1). Choosing one such diagram, it follows from surjectivity of  $g$  that the restriction-and-pull-back map

$$H^0(Z, \Omega_Z^{[p]}) \rightarrow H^0(V, \Omega_V^p), \quad \mu \mapsto d_{\text{Kähler}} g(\mu|_{Z_{\text{reg}}})$$

is injective. Equation (6.1.2) therefore determines the reflexive form  $\tau$  uniquely. This finishes the proof of Proposition 6.1  $\square$

## 7. PROOF OF THEOREM 5.2

**7.1. Proof of Theorem 5.2: Uniqueness of the functor.** Assume we are given two functors  $d_{\text{refl}}^1$  and  $d_{\text{refl}}^2$  which satisfy the conditions of Theorem 5.2. Given any morphism of klt base spaces,  $f : Z \rightarrow X$ , denote the two associated pull-back maps of reflexive differentials by

$$d_{\text{refl}}^1 f : H^0(X, \Omega_X^{[p]}) \rightarrow H^0(Y, \Omega_Y^{[p]}) \quad \text{and} \quad d_{\text{refl}}^2 f : H^0(X, \Omega_X^{[p]}) \rightarrow H^0(Y, \Omega_Y^{[p]}),$$

respectively. To prove Theorem 5.2, we need to show that

$$(7.1.1) \quad d_{\text{refl}}^1 f = d_{\text{refl}}^2 f.$$

This equality will be established in the remainder of the present Section 7.1.

**7.1.1. Proof of uniqueness in special cases.** Before proving Equality (7.1.1) in general, we treat two special cases first.

**Lemma 7.2.** *If there exists a dense open subset  $Z^\circ \subseteq Z$  with inclusion  $i : Z^\circ \rightarrow Z$  such that*

$$(7.2.1) \quad d_{\text{refl}}^1(f \circ i) = d_{\text{refl}}^2(f \circ i),$$

*then Equation (7.1.1) holds.*

*Proof.* By functoriality, Assumption (7.2.1) translates as

$$d_{\text{refl}}^1 i \circ d_{\text{refl}}^1 f = d_{\text{refl}}^2 i \circ d_{\text{refl}}^2 f.$$

To prove Lemma 7.2, it will therefore suffice to show that  $d_{\text{refl}}^1 i = d_{\text{refl}}^2 i$ , and that these maps are injective. That, however, follows immediately from compatibility with Kähler differentials, as formulated in Diagram (5.2.2) of Theorem 5.2: given any reflexive form  $\sigma \in H^0(Z, \Omega_Z^{[p]})$ , then

$$(d_{\text{refl}}^1 i)(\sigma)|_{Z_{\text{reg}}^\circ} = (d_{\text{Kähler}} i)(\sigma|_{Z_{\text{reg}}^\circ}) = (d_{\text{refl}}^2 i)(\sigma)|_{Z_{\text{reg}}^\circ}.$$

Observe that this determines  $(d_{\text{refl}}^\bullet i)(\sigma)$  uniquely. □

**Lemma 7.3.** *If  $f$  is surjective, then Equation (7.1.1) holds.*

*Proof.* It is clear by assumption that the open set  $Z^\circ := Z_{\text{reg}} \cap f^{-1}(X_{\text{reg}}) \subseteq Z$  is not empty. Compatibility with Kähler differentials then asserts that  $d_{\text{refl}}^1(f \circ i) = d_{\text{refl}}^2(f \circ i)$ , where  $i : Z^\circ \rightarrow Z$  is the obvious open immersion. Lemma 7.2 therefore applies. □

**7.1.2. Proof of uniqueness in general.** Next, we treat the general case. To this end, consider a commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \pi_Z \downarrow & & \downarrow \pi \\ Z & \xrightarrow{f} & X \end{array} \quad \begin{array}{l} \text{strong resolution of} \\ \text{singularities} \end{array}$$

where  $\tilde{Z}$  is a desingularization of a component of the fibre product  $Z \times_X \tilde{X}$  which surjects onto  $Z$ . Using Lemma 7.2, we may replace  $Z$  with a dense open subset

and assume without loss of generality that  $Z$  is smooth. The following identities are now immediate consequences of compatibility with Kähler differentials.

$$(7.4.1) \quad d_{\text{refl}}^1 \pi = d_{\text{refl}}^2 \pi \quad \text{by Lemma 7.3}$$

$$(7.4.2) \quad d_{\text{refl}}^1 \tilde{f} = d_{\text{refl}}^2 \tilde{f} = d_{\text{Kähler}} \tilde{f} \quad \text{by (5.2.2) since } \tilde{Z} \text{ and } \tilde{X} \text{ are smooth}$$

$$(7.4.3) \quad d_{\text{refl}}^1 \pi_Z = d_{\text{refl}}^2 \pi_Z = d_{\text{Kähler}} \pi_Z \quad \text{by (5.2.2) since } \tilde{Z} \text{ and } Z \text{ are smooth}$$

The equalities have several consequences. First, we see that

$$\begin{aligned} d_{\text{refl}}^1(\pi \circ \tilde{f}) &= (d_{\text{refl}}^1 \tilde{f}) \circ (d_{\text{refl}}^1 \pi) && \text{functoriality of } d_{\text{refl}}^1 \\ &= (d_{\text{refl}}^2 \tilde{f}) \circ (d_{\text{refl}}^2 \pi) && \text{by (7.4.1), (7.4.2)} \\ &= d_{\text{refl}}^2(\pi \circ \tilde{f}) && \text{functoriality of } d_{\text{refl}}^2. \end{aligned}$$

Since  $\pi \circ \tilde{f} = f \circ \tilde{\pi}_Z$ , this immediately implies that

$$(7.4.4) \quad (d_{\text{refl}}^1 \pi_Z) \circ (d_{\text{refl}}^1 f) = (d_{\text{refl}}^2 \pi_Z) \circ (d_{\text{refl}}^2 f)$$

But since  $d_{\text{refl}}^1 \pi_Z$  and  $d_{\text{refl}}^2 \pi_Z$  are both equal to the standard pull-back of Kähler differentials, and since  $\pi_Z$  is surjective, it is clear that two forms  $\sigma, \tau$  on  $Z$  agree if and only if  $d_{\text{refl}}^1 \pi_Z(\sigma) = d_{\text{refl}}^2 \pi_Z(\tau)$ . Equation (7.4.4) therefore implies Assertion (7.1.1), finishing the proof of the uniqueness statement in Theorem 5.2.  $\square$

**7.2. Proof of Theorem 5.2: Existence of the functor.** Given a morphism  $f : Z \rightarrow X$  of klt base spaces, we define an associated pull-back mapping of reflexive differential forms,

$$d_{\text{refl}} f : H^0(X, \Omega_X^{[p]}) \rightarrow H^0(Z, \Omega_Z^{[p]}),$$

by sending a given reflexive form  $\sigma \in H^0(X, \Omega_X^{[p]})$  to the unique form  $\tau \in H^0(Z, \Omega_Z^{[p]})$  whose existence is asserted in Proposition 6.1. We need to show that the so-defined  $d_{\text{refl}}$  satisfies the composition law. To this end, consider a sequence of morphisms between klt base spaces, say  $f' : Z' \rightarrow Z$  and  $f : Z \rightarrow X$ . We need to show that

$$(7.5.1) \quad (d_{\text{refl}} f' \circ d_{\text{refl}} f)(\sigma) \stackrel{!}{=} (d_{\text{refl}}(f \circ f'))(\sigma) \quad \text{for all } \sigma \in H^0(X, \Omega_X^{[p]}).$$

To prove (7.5.1), choose desingularisation  $p : \tilde{Z} \rightarrow Z$ , choose a component of  $\tilde{Z} \times_X \tilde{X}$  that surjects onto  $\tilde{Z}$ , and let  $\hat{Z}$  be a desingularisation of that component. Further, choose a component of  $\hat{Z} \times_Z Z'$  that surjects onto  $Z'$ , and let  $V$  be a desingularisation of that component. In summary, we have constructed smooth spaces  $\tilde{Z}, \hat{Z}$  and  $V$  fitting into a commutative diagram

$$(7.5.2) \quad \begin{array}{ccccc} & & \hat{Z} & \xrightarrow{h} & \tilde{X} \\ & \nearrow \hat{a} & \downarrow q \text{ surjective} & & \downarrow \pi \text{ resolution of singularities} \\ & & \tilde{Z} & & \\ & \nearrow a & \downarrow p \text{ resolution of singularities} & & \\ V & \xrightarrow{g, \text{ surjective}} & Z' & \xrightarrow{f'} & Z & \xrightarrow{f} & X. \end{array}$$

The following reflexive forms, defined as explained above using Proposition 6.1, will appear in the computation

$$\tilde{\sigma} := d_{\text{refl}} \pi(\sigma) \quad \tau := d_{\text{refl}} f(\sigma) \quad \tilde{\tau} := d_{\text{refl}} p(\tau).$$



We have seen in Remark 6.3 that this definition implies that  $\tilde{\sigma}$  is the unique differential form on  $\tilde{X}$  that agrees with the pull-back  $\sigma$  at points where  $\pi$  is isomorphic. The analogue statement holds for  $\tilde{\tau}$  and  $\tau$ , so that our notation is consistent with the notation used earlier. A repeated application of Proposition 6.1, using the fact that two reflexive forms on a normal space agree if they agree on the smooth locus, now shows the following

$$\begin{aligned}
 d_{\text{refl}}g((d_{\text{refl}}f' \circ d_{\text{refl}}f)(\sigma)) &= d_{\text{refl}}g(d_{\text{refl}}f'(\tau)) && \text{Definition of } \tau \\
 &= d_{\text{refl}}a(\tilde{\tau}) && \text{Proposition 6.1 for } f' \\
 &= (d_{\text{refl}}\hat{a} \circ d_{\text{refl}}q)(\tilde{\tau}) && \text{Remark 6.3, Diag. (7.5.2)} \\
 &= (d_{\text{refl}}\hat{a} \circ d_{\text{refl}}h)(\tilde{\sigma}) && \text{Proposition 6.1 for } f \\
 &= d_{\text{refl}}g(d_{\text{refl}}(f \circ f')(\tilde{\sigma})) && \text{Proposition 6.1 for } f \circ f'.
 \end{aligned}$$

Since  $g$  is surjective, Remark 6.3 immediately implies that  $d_{\text{refl}}g$  is injective. Equation (7.5.1) therefore follows from the computation. In summary, we have shown that the definition of pull-back given above does satisfy the composition law. This finishes the proof of the existence statement in Theorem 5.2.  $\square$

### Part III. Appendix

#### APPENDIX A. TORSION SHEAVES ON REDUCIBLE SPACES

In Parts I and II of this paper, we need to discuss torsion sheaves and torsion-free sheaves on reducible spaces. While no fundamental issues arise, it seems that almost all standard books, such as [Har77], [Gro60] or [GR84] restrict themselves to the reducible case. The few existing references touch the subject only very briefly. For completeness' sake, we have thus chosen to recall the relevant definitions and to include proofs of all the properties used in this paper.

**A.1. The definition of torsion sheaves.** We briefly recall the definition of torsion sheaves given in [DG71, I.8]<sup>4</sup>, see also [Gro67, §20.1].

*Notation A.1* (Sheaf of rational functions, [DG71, I.8.3]). Let  $X$  be a reduced, quasi-projective scheme. We denote the sheaf of rational functions on  $X$  by  $\mathcal{R}_X$ .

*Explanation A.2.* In the setting of Notation A.1, the sheaf of rational functions is quasi-coherent. If  $X^\circ \subseteq X$  is an affine open set, say  $X^\circ = \text{Spec } A$ , then  $\mathcal{R}_X(X^\circ)$  is the ring of rational functions on  $X$ . This ring is isomorphic to the localisation  $S^{-1}A$ , where  $S$  is the multiplicatively closed set of non-zero-divisors in  $A$ .

**Definition A.3** (Torsion sheaf, [DG71, I.8.4]). Let  $X$  be a reduced, quasi-projective scheme and  $\psi : \mathcal{O}_X \rightarrow \mathcal{R}_X$  the natural inclusion of the structure sheaf into the sheaf of rational functions. Given a coherent sheaf  $\mathcal{F}$  of  $\mathcal{O}_X$ -modules consider the natural map  $\psi_{\mathcal{F}}$  given as the composition of the following maps,

$$\mathcal{F} \xrightarrow{\cong} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \xrightarrow{\text{Id}_{\mathcal{F}} \otimes \psi} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}_X.$$

Define the torsion subsheaf of  $\mathcal{F}$  as  $\text{tor } \mathcal{F} := \ker \psi_{\mathcal{F}}$ . The sheaf  $\mathcal{F}$  is called torsion sheaf if  $\psi_{\mathcal{F}} = 0$ , and torsion-free if  $\psi_{\mathcal{F}}$  is injective.

*Explanation A.4.* In the setting of Definition A.3, let  $X^\circ \subseteq X$  be any affine open set, say  $X^\circ = \text{Spec } A$ . Denoting the  $A$ -module associated with the sheaf  $\mathcal{F}$  by  $F := \mathcal{F}(X^\circ)$ , the sheaf  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}_X$  of Definition A.3 is expressed as follows,

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}_X = \mathcal{F}^\sim \otimes_{\mathcal{O}_X} (S^{-1}A)^\sim = (F \otimes_A S^{-1}A)^\sim = (S^{-1}F)^\sim.$$

<sup>4</sup>The definition presented here is found in [DG71] but not in [Gro60]. At the time of writing this paper, the book [DG71] was not listed on MathSciNet and did not show on [www.springer.com](http://www.springer.com).

In summary, we see that a section  $\sigma \in \mathcal{F}(X^\circ)$  is a section of the torsion subsheaf  $\text{tor } \mathcal{F}$  if and only if there exists a non-zero-divisor  $f \in \mathcal{O}_X(X^\circ)$  which annihilates it. In particular,  $\sigma$  is a section of the torsion subsheaf if and only if there exists dense open  $U \subseteq X^\circ$  such that  $\sigma|_U = 0$ .

*Remark A.5* (Torsion-free sheaves that are zero on open sets). Torsion-free sheaves on reducible spaces can restrict to the zero sheaf on a Zariski-open set, as long as the set is not dense.

**A.2. Elementary properties.** Definition A.3 ensures that essentially all properties known from torsion-free sheaves on irreducible spaces also hold in the more general setting of reduced quasi-projective schemes—except perhaps the feature mentioned in Remark A.5 above. The following properties have been used in this paper.

**Proposition A.6** (Universal property of torsion-freeness). *Let  $X$  be a reduced quasi-projective scheme, and  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  a morphism of coherent sheaves of  $\mathcal{O}_X$ -modules. Then there exists a unique morphism  $\check{\phi}$  making the following diagram commutative,*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ \text{quotient} \downarrow & & \downarrow \text{quotient} \\ \mathcal{F}/\text{tor} & \xrightarrow{\check{\phi}} & \mathcal{G}/\text{tor}. \end{array}$$

*If  $\psi$  is injective, then  $\check{\phi}$  is injective as well.*

*Proof.* Observing that  $\mathcal{F}/\text{tor} = \text{Image } \psi_{\mathcal{F}}$  and  $\mathcal{G}/\text{tor} = \text{Image } \psi_{\mathcal{G}}$ , the claim is immediate from right-exactness and from universal properties of the tensor product.  $\square$

**Corollary A.7** (Subsheaves of torsion-free sheaves are torsion-free). *Let  $X$  be a reduced quasi-projective scheme, and  $\mathcal{F}$  a torsion-free coherent sheaf of  $\mathcal{O}_X$ -modules. If  $\mathcal{G}$  is any subsheaf of  $\mathcal{O}_X$ -modules, then  $\mathcal{G}$  is likewise torsion-free.*  $\square$

**Proposition A.8** (Push-forward of torsion-free is torsion-free, compare [DG71, I. Prop. 8.4.5]). *Let  $X$  be a reduced quasi-projective scheme. If  $\iota : X_0 \rightarrow X$  denotes the inclusion map of one irreducible component, and if  $\mathcal{F}$  is a torsion-free coherent sheaf on  $X_0$ , then  $\iota_*\mathcal{F}$  is a torsion-free coherent sheaf on  $X$ .*

*Proof.* Let  $X^\circ \subset X$  be any affine open subset. If  $X^\circ$  is disjoint from  $X_0$ , then  $\iota_*\mathcal{F}|_{X^\circ}$  is the zero sheaf, which is torsion-free.

Now assume that  $X^\circ \cap X_0 \neq \emptyset$ , and let

$$\sigma \in (\text{tor } \iota_*\mathcal{F})(X^\circ) \subseteq (\iota_*\mathcal{F})(X^\circ)$$

be any section, with associated section  $\tau \in \mathcal{F}(X^\circ \cap X_0)$ . We need to show that  $\sigma = 0$ , or equivalently  $\tau = 0$ . To this end, let  $X_1 \subset X$  denote the union of all irreducible components different from  $X_0$ . Let  $U \subset X^\circ$  be a dense open subset that that  $\sigma|_U = 0$ . The set  $U^\circ := U \cap (X_0 \setminus X_1)$  is then open and dense on  $X_0$  and  $\sigma|_{U^\circ} = 0$ . In particular,  $\tau|_{U^\circ} = 0$ . Since  $\mathcal{F}$  is torsion-free, this shows that  $\tau = 0$ , as claimed.  $\square$

**Proposition A.9** (Injectivity of morphisms). *Let  $X$  be a reduced, quasi-projective scheme and  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  a morphism of coherent sheaves of  $\mathcal{O}_X$ -modules. Assume that  $\mathcal{F}$  is torsion-free. If  $Y \subsetneq X$  is a closed subset such that*

(A.9.1)  *$Y$  does not contain any irreducible component of  $X$ , and*

(A.9.2) *the restricted morphism  $\phi|_{X \setminus Y}$  is injective,*

then  $\phi$  is injective.

*Proof.* Assume we are given an affine open set  $X^\circ \subseteq X$  and a section  $\sigma \in \ker \phi$ . Observe that  $U := X^\circ \setminus Y$  is dense in  $X^\circ$ , and that  $\sigma|_U = 0$ . Since  $\mathcal{F}$  is torsion-free this implies that  $\sigma = 0$ .  $\square$

## REFERENCES

- [DG71] Jean A. Dieudonné and Alexandre Grothendieck, *Éléments de Géométrie Algébrique. I.*, Die Grundlehren der mathematischen Wissenschaften, Band 166, Springer-Verlag, Berlin-Heidelberg-New York, 1971. [↑ 30, 31](#)
- [EV92] Hélène Esnault and Eckart Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR MR1193913 (94a:14017) [↑ 13](#)
- [GHS03] Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic). MR 1 937 199 [↑ 5](#)
- [GKK10] Daniel Greb, Stefan Kebekus, and Sándor J Kovács, *Extension theorems for differential forms, and Bogomolov-Sommese vanishing on log canonical varieties*, Compositio Math. **146** (2010), 193–219, DOI:10.1112/S0010437X09004321. A slightly extended version is available as [arXiv:0808.3647](#). [↑ 26](#)
- [GKKP11] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell, *Differential forms on log canonical spaces*, Inst. Hautes Études Sci. Publ. Math. **114** (2011), no. 1, 87–169, DOI:10.1007/s10240-011-0036-0 An extended version with additional graphics is available as [arXiv:1003.2913](#). [↑ 2, 11, 16, 21, 22, 23, 24](#)
- [GKP11] Daniel Greb, Stefan Kebekus, and Thomas Peternell, *Singular spaces with trivial canonical class*, preprint [arXiv:1110.5250](#)., October 2011. [↑ 2, 16](#)
- [GR84] Hans Grauert and Reinhold Remmert, *Coherent analytic sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 265, Springer-Verlag, Berlin, 1984. MR 755331 (86a:32001) [↑ 30](#)
- [GR11] Daniel Greb and Sönke Rollenske, *Torsion and cotorsion in the sheaf of Kähler differentials on some mild singularities*, Math. Res. Lett. **18** (2011), no. 6, 1–11. [↑ 15](#)
- [Gro60] Alexandre Grothendieck, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. (1960), no. 4, 228. MR 0217083 (36 #177a) [↑ 30](#)
- [Gro67] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 0238860 (39 #220) [↑ 30](#)
- [Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116) [↑ 14, 30](#)
- [HM07] Christopher D. Hacon and James McKernan, *On Shokurov’s rational connectedness conjecture*, Duke Math. J. **138** (2007), no. 1, 119–136. MR 2309156 (2008f:14030) [↑ 5, 25](#)
- [Keb11] Stefan Kebekus, *Differential forms on singular spaces, the minimal model program, and hyperbolicity of moduli stacks*, preprint [arXiv:1107.4239](#). To appear in the “Handbook of Moduli, in honour of David Mumford”, to be published by International press, editors Gavril Farkas and Ian Morrison, July 2011. [↑ 2](#)
- [KK10] Stefan Kebekus and Sándor J. Kovács, *The structure of surfaces and threefolds mapping to the moduli stack of canonically polarized varieties*, Duke Math. J. **155** (2010), no. 1, 1–33, preprint [arXiv:0707.2054](#). MR 2730371 (2011i:14060) [↑ 2](#)
- [KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. MR 2000b:14018 [↑ 11](#)
- [Kol12] János Kollár, *Deformations of elliptic Calabi–Yau manifolds*, preprint [arXiv:1206.5721](#)., June 2012. [↑ 2](#)
- [Nam01a] Yoshinori Namikawa, *Deformation theory of singular symplectic  $n$ -folds*, Math. Ann. **319** (2001), no. 3, 597–623. MR 1819886 (2002b:32025) [↑ 25](#)
- [Nam01b] ———, *Extension of 2-forms and symplectic varieties*, J. Reine Angew. Math. **539** (2001), 123–147. MR 1863856 (2002i:32011) [↑ 12](#)
- [VZ02] Eckart Viehweg and Kang Zuo, *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328. MR 1922109 (2003h:14019) [↑ 11](#)

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